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A Way to Construct a Hybrid Forward-Jumping Method

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Abstract. One of the main problems in the theory of numerical methods is the construction of the methods with the high accuracy and having extended stability regions. Depending from the object of investigation, it usually requires the necessary of some additional requirements on the using methods as a decreasing ofvolume computational work, the using of the certain properties of the solutions of the investigated problem, the selection of the step size, the determination of the values of the considering problem at intermediate points, etc. Therefore, the method which has the widely application must satisfy certain additional conditions. As a rule, the methods have constructed by the above mentioned requirements do not have high accuracy. Therefore, the methods specially constructed to solve specific problems, are more effective.By using the above presentationhere we are consider to the construction of a special method and its application to solving of the initial value problem for ODE of the first order.

1. Introduction

Let us consider, we investigate the finding of a numerical solution of the following initial problem:

$$y' = f(x, y), \quad y(x_0) = y_0, \ x_0 \le x \le X . \ (1)$$

Assume that the initial value problem (1) has a continuous unique solution defined on the segment $[x_0, X]$. To find an approximate value of the solution of problem (1), the segment $[x_0, X]$ is divided into N equal parts by the step size $h > 0$, and the mesh point is defined as $x_i = x_0 + ih \ (i = 0,1,..., N)$. In addition, we denote the approximate values of the solution of problem (1) by y_m at the mesh point $x_m \ (m = 0,1,...)$ but by the $y(x_m)$ corresponding exact values of the solution of the problem (1).

The many known scientists have been investigated the numerical solutions of problem (1), beginning from the Clairaut (see, for example, [1, p.132]). However, as noted in [2, p.289], the first direct numerical method for solving of problem (1) was constructed by Euler. Consequently, the numerical solution of problem (1) has been studied for a long time. But, the construction of numerical methods for solving of the problem (1), having higher accuracy and an extended stability region, is actual at the present time.

As is known, scientists from different countries to obtain more accurate results in solving practical problems, considered the generalization of the Euler method, in resulting which the emergence of one and multi-step methods.

The known representatives of these methods are the Runge-Kutta and Adams methods, each of which has the advantages and disadvantages. Runge-Kutta methods have been fundamentally investigated by J. Butcher (see [3]).J. Butcher investigated the explicit, semi-explicit and implicit Runge-Kutta methods and constructed formulas for determining the maximum values of accuracy for the Runge-Kutta methods. Note that the many scientists are developed Runge-Kutta methods (see e.g. [4] - [9]).
The aim of this paper is to construct the methods on the junction of the forward jumping and hybrid methods. These methods are multi-step and belong to the class of multistep methods with the constant coefficients.
Multistep methods have been fundamentally investigated by Dahlquist (see [10] - [12]), the generalization of which is usually called the k-step methods of Obreshkov type or multistep multi derivative methods that were fundamentally investigated in [13] (see also [14]).
As is known, the forward-jumping methods are not coincide with the Adams methods and are not enter to the class of multistep Obreshkov'smethods. However, the accuracy for known stable methods of forward-jumping type is subordinates to Dahlquist's law, which was the main obstacle in the development of the forward-jumping methods. In [1] provethe existence of stable forward-jumping methods with the order of accuracy $p > 2[k/2]+2$. And for the use of such methods have been constructed special predictor-corrector methods (see [13] - [15]). And also shown, that by the help of the choosing the predictor methods it is possible to construct the predictor-corrector methods by using forward jumping methods with the extended stability region (see e.g. [6]).
Hybrid multistep methods, proposed by Gear and Butcher (see [17], [18]), are intensively studied in recent years. There are many works dedicated to their advantages, which make it possible to extendthe field of their applications. Therefore, here we consider the construction of the forward-jumping methods of a hybrid type (see e.g. [19] - [26]).
As noted, one of the popular methods for solving problem (1) is a multi-step method with the constant coefficients having the following form:

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i f_{n+i}, \quad n = 0,1,...,N-k \ . \qquad (2)$$

It is easy to understand that the method (2) is a generalization of Adams' methods, which has been studied by many specialists from different countries. Here we assume that the coefficients of method (2) satisfy the conditions A, B and C from the work of Dahlquist (see [10]). In [10] has been proved that if the method (2) is stable, then $p \le 2[k/2]+2$. Here the integer valued quantity p is the degree and the integer valued quantity k is the order of the method (2).
For the construction of more precise methods, it is possible to use the hybrid methods, which in more general form can be written as the following:

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i f_{n+i} + h\sum_{i=0}^{k}\gamma_i f_{n+i+\nu_i} \quad \left(|\nu_i| < 1;\ i = 0,1,...,k\right) \ . \qquad (3)$$

This method has been investigated by many authors (see e.g. [24], [26], [29], [30].

2. One way for constructing of the hybrid forward-jumping methods.

For the construction of hybrid forward-jumping methods is proposed to use the following formula:

$$\sum_{i=0}^{k-m}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i f_{n+i} + h\sum_{i=0}^{k}\gamma_i f_{n+i+\nu_i} \quad \left(m > 0;\ |\nu_i| < 1;\ i = 0,1,...,k\right) \ . (4)$$

Method (4) was constructed taking into account the fact that in the class of the methods (5), there exist stable methods with the degree $p = k+m+1 \ \ (k \ge 3m)$, and there exist stable methods with degree $p = 2k+2$ in the class of methods of type (3) (see [14]) .Therefore, method (4) was constructed at the junction of the forward jumping and hybrid methods. Here we try to choose the quantities $\alpha_i \ \ (i = 0,1,...,k-m), \beta_i, \gamma_i, \nu_i \ (i = 0,1,...,k)$ so that the methods obtained from formula (4) have high accuracy and the extended stability region.

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In order to study the methods (4), we assume that the coefficients α_i $(i=0,1,...,k-m), \beta_i$, γ_i $(i=0,1,...,k)$ satisfies the following conditions:

A: The coefficients α_i $(i=0,1,2,...,k-m)$, β_i, γ_i, ν_i $(i=0,1,2,...,k)$ are some real numbers, moreover, $\alpha_k \neq 0$.

B: Characteristic polynomials

$$\rho(\lambda) \equiv \sum_{i=0}^{k-m} \alpha_i \lambda^i, \quad \sigma(\lambda) \equiv \sum_{i=0}^{k} \beta_i \lambda^i; \ \gamma(\lambda) \equiv \sum_{i=0}^{k} \gamma_i \lambda^{i+\nu_i}.$$

have no common multipliers different from the constant.

C: $\sigma(1) + \gamma(1) \neq 0$ and $p \geq 1$.

For the purpose of constructing a system of algebraic equations, to find the coefficients of the method (9) $\alpha_i (i=0,1,...,k-m), \beta_i, \gamma_i, \nu_i$ $(i=0,1,...,k)$,we use the method of undetermined coefficients (see e.g. [24]), in the results of which receive a system of nonlinear algebraic equations, which for I $m=1$ u $k=2$ can be written as the following:

$$\begin{aligned} &\beta_2 + \beta_1 + \beta_0 + \gamma_2 + \gamma_1 + \gamma_0 = 1, \\ &2^j \beta_2 + \beta_1 + l_2^j \gamma_2 + l_1^j \gamma_1 + l_0^j \gamma_0 = 1/(j+1), (j = 1,2,...,9) \end{aligned} \quad (5)$$

Note that for $m=1$ and $k=2$ the unknowns following quantities are considered: $\alpha_0, \alpha_1, \beta_0, \beta_1, \beta_2, \gamma_0, \gamma_1, \gamma_2, \nu_0, \nu_1, \nu_2$. However, from the necessary condition for the convergence of method (9), we obtain that $\rho(1) = 0$. Consequently $\alpha_1 + \alpha_0 = 0$. It follows that $\alpha_1 = -\alpha_0 = 1$.

Using solutions of system (5) for $j=9$, one can construct the stable methods of type (9) having the degree $p \leq 9$. It does not follow from this that there are no stable methods with the degree $p > 9$ at $m=1$ and $k=2$. To simplify the system (5), let us consider the case $\nu_1 = 0$; $\nu_0 = 1/2$; $\nu_1 = 0$ $\nu_2 = -1/2$; and $\gamma_1 = \beta_1$. Then, by solving the system (5) we obtain that.

$$\beta_0 = 29/180; \beta_1 = 6/90; \beta_2 = -1/180; \ \gamma_0 = 21/45; \gamma_1 = 6/90; \gamma_2 = 1/45.$$

By using the obtained solution in the formula (4), one can be obtain the following hybrid forward-jumping method:

$$y_{n+1} = y_n + h(29 f_n + 24 f_{n+1} - f_{n+2})/180 + h(62 f_{n+1/2} + 2 f_{n+3/2})/90. \quad (6)$$

The local error, which can be represented in the form:

$$-\frac{h^6}{720} y^{(6)}(x) \Big|_{x=x_n} + O(h^7).$$

Considering that the method (6) is constructed at the junction of the forward jumping and the hybrid methods; let us compare the method (6) with the following hybrid and forward jumping methods:

$$y_{n+1} = y_n + h f_n / 9 + h\Big((16+\sqrt{6}) f_{n+(6-\sqrt{6})/10} + (16-\sqrt{6}) f_{n+(6+\sqrt{6})/10}\Big)/36 , \quad (7)$$

$$y_{n+2} = (8 y_{n+1} + 11 y_n)/19 + h(10 f_n + 57 f_{n+1} + 24 \hat{f}_{n+2} - f_{n+3})/57. \quad (8)$$

These methods are stable and have the degree $p = 5$.

Remark, that the forward-jumping method has some advantages. However, for using them, becomes necessary to determine the values of the desired functions at subsequent points, which are the main disadvantages of the forward-jumpingmethods. But, if we use the predictor-corrector method to find these values, then all these methods become almost equivalent from the point of view of computational work. In our case, in the method (6) we replace the step of the integration h by $h/2$, as a result of which we obtain:

$$y_{n+1} = (11 y_n + 8 y_{n+1/2})/19 + h(10 f_n + 57 f_{n+1/2} + 24 f_{n+1} - f_{n+3/2})/144. \quad (9)$$

Note that this method is not a particular case of the above mentioned methods, but remained a method of the type (4). Now consider to the construction of the predictor-corrector method for using the method (9). To this $y_{n+1/2}$ end, we propose to use the following methods, if the approximate value of the quantity is known:

$$\hat{y}_{n+3/2} = y_{n+1} + h(23 f_{n+1} - 16 f_{n+1/2} + 5 f_n)/24, \qquad (10)$$

$$y_{n+3/2} = y_{n+1} + h(9\hat{f}_{n+3/2} + 19 f_{n+1} - 5 f_{n+1/2} + f_n)/48. \quad (11)$$

Since method (11) has the degree $p = 4$, therefore when used that in the linear part of the method (9) the error of the received method will be has the degree less than the degree of the method (9). Therefore, it is desirable to use the methods (10) and (11) together with the method (8). And as the predictor method for calculation the value $\hat{y}_{n+2}$, one can use the Simpson method, which is stable and has the degree $p = 4$. However, to change (8) by the method (9) increases its accuracy, but complicates its application to solving of a model problem.

3. An illustration of the obtainedresults

To illustrate the results obtained, here we consider the application of the constructed numerical method (6) to solving of the model problem. This approach is justified by the fact that the model problem correctly describes the solution for many applied problems and coincides with the main part in the asymptotic expansion of the solutions of many scientific and technical problems. It should be noted that many scientific papers prefer to compare methods by using the model problem.
Consider the solution of the following problem

$y' = \lambda y,\ y(0) = 1,\quad 0 \le x \le 1$ (exact solution: $y(x) = \exp(x)$).

For solving of this problem, we have used the methods (6) and (8) with the step $h = 0{,}01$ and for the values $\lambda = \pm 1; \pm 5$. For compiling algorithms, the Simpson method is used as a predictor method. The results are placed in the following table:

λ	x_n	Error for method (6)	Error for method (8)
		$h = 0.1$	
$\lambda = 1$	0.1	$4.66E-12$	$1.28E-10$
	0.4	$2.96E-11$	$7.27E-10$
	0.7	$7.15E-11$	$1.72E-9$
	1.0	$1.39E-10$	$3.34E-9$
$\lambda = -1$	0.1	$3.86E-12$	$1.04E-10$
	0.4	$1.34E-11$	$3.24E-10$
	0.7	$1.78E-11$	$4.23E-10$
	1.0	$1.90E-11$	$4.49E-10$
$\lambda = 5$	0.1	$2.12E-8$	$4.23E-8$
	0.4	$4.48E-7$	$3.96E-8$
	0.7	$3.59E-6$	$1.55E-8$
	1.0	$2.31E-5$	$4.98E-9$
$\lambda = -5$	0.1	$8.27E-9$	$1.19E-7$
	0.4	$8.69E-9$	$2.24E-6$
	0.7	$3.46E-9$	$1.77E-5$
	1.0	$1.11E-9$	$1.13E-4$

Note that, the results obtained by using the methods, constructer in junction of the forward jumping and hybrid methods are better than the results obtained by using the forward-jumping methods. However, if in the algorithm using method (11), as the predictor method, the midpoint method was proposed, then the result deteriorates .In the following table, we haveplaced the results obtained by using the methods (11), the midpoint method and the following hybrid method

$$y_{n+1} = y_n + h(f_{n+\alpha} + f_{n+1-\alpha})/2 \ . \qquad (12)$$

λ	x_n	Error for method (6)	Error for method (12)
$h = 0.1$			
$\lambda = 1$	0.1 0.4 0.7 1.0	$3.16E-9$ $1.99E-8$ $4.8E-8$ $9.3E-8$	$4.56E-9$ $2.46E-8$ $5.82E-8$ $1.12E-7$
$\lambda = -1$	0.1 0.4 0.7 1.0	$2.62E-9$ $9.06E-9$ $1.19E-8$ $1.27E-8$	$3.80E-9$ $1.12E-8$ $1.46E-8$ $1.54E-8$
$\lambda = 5$	0.1 0.4 0.7 1.0	$2.86E-6$ $6.00E-5$ $4.80E-4$ $3.10E-3$	$4.12E-6$ $7.39E-5$ $5.80E-4$ $3.71E-3$
$\lambda = -5$	0.1 0.4 0.7 1.0	$1.12E-6$ $1.17E-6$ $4.67E-7$ $1.50E-7$	$1.64E-6$ $1.46E-6$ $5.72E-7$ $1.82E-7$

4. Conclusions

For finding a numerical solution of the problem (1), by means of comparison of some different methods we have constructed a new method and showed its advantages. The proposed method can be considered promising, because that uses the best qualities of hybrid and forward-jumping methods. To illustrate these qualities, the order of increasing and decreasing the solution for the model problem was used. As can be seen from the above tables, the results obtained by using the methods proposed here are the best in all the cases. As is known, when applying implicit and hybrid methods to solving some practical problems, it becomes necessary to use predictor methods, from which depends the order of accuracy of received results. To confirm this, one can by use the results placed in the above mentioned tables.

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