Dynamic Interaction of Interfacial Point Source Loading and Cylinder in an Elastic Quarter with Anti-plane Shear

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Abstract. Theoretical steady state solution of a semi-circular cylinder impacted by an anti-plane point loading in a vertical bound of an elastic quarter is formulated in this paper through using image method and wave function expansion series. The elastic quarter is extended as a half space, and the semi-circular interfacial cylinder is extended as a circular cylinder. Displacement field is constructed as series of Fourier-Hankel and Fourier-Bessel wave functions. At last, circular boundary is expanded as Fourier series to determine coefficients of wave function. Numerical results show that material parameters have two widely divergent effects on the radial and circumferential dynamic stress distribution.

1. Introduction

As the simplest elastodynamics research object, anti-plane movement has an important theoretical significance. As the basic theory of anti-plane elastodynamics theory, SH wave propagation and scattering have a wide application in earthquake engineering, geotechnical engineering, material engineering and so on. In the early 1980s, Pao Y H, Mow C C[1] conducted systematic studied the SH wave scattering of circular cylinder. Since this century, Liu D K[2],Shi S X [3] Tian J Y[4,5] and Qi H[6-8] studied the scattering of plane SH wave and anti-plane line source load which influenced by interface circular cavity or cylinder and lining structure in full space. Qi H [9], Yang Z L[10] studied the interaction of convex structure with different foundation under the incidence of SH wave. In recent years, Qi H, Zh Y[11-13] ,Yang J[14-16] studied the scattering of SH wave and point source load which influenced by cylinder cavity or inclusion in elastic quarter space and bi-material. And they made certain achievements. But the research of anti-plane scattering of interface cylinder in the bi-material half space is still insufficient. In this paper, the steady state of interface cylinder which subjected by anti-plane point source load in elastic quarter space was studied. This is the basis to formulate Green function method for SH wave scattering of bi-material interface circular cylinder. Using the steady state solution of the point source load and superposition principle, the steady state response of an arbitrary distributed interface load to an elastic quarter space of interface semi-circular cylinder can be got. At the same time, through numerical calculating examples above, the conclusion can provide certain reference for theoretical research and engineering practice.

2. Descriptions and analysis

In this paper, we study anti-plane elastodynamics problems. Figure 1 shows an elastic quarter space. The horizontal boundary is plane B_H , vertical boundary is plane B_V . Embedded in a semi-circular cylinder, the center point O is located in plane B_V on the vertical boundary, and the direct distance between horizontal boundary plane B_H is h. The shearing elasticity modulus of the quarter space is μ_1, mass density is ρ_1. The shearing elasticity modulus of semi-circular cylinder is μ_2, mass density is ρ_2. Build a plane rectangular coordinate system (O,x,y), let the center point O of semi-circular cylinder as the origin, inner normal orientation of vertical boundary plane B_V as positive direction of axis x, and outer normal orientation of horizontal boundary plane B_H as positive direction of axis y. Taking the origin point O as the pole, and axis x as the polar axis, the polar coordinate system (O,r,θ)) is set up，according to the counterclockwise direction. Out-of-plane orientation（"⦿"in figure1） as positive direction of anti-plane displacement, and in-plane orientation as negative direction.

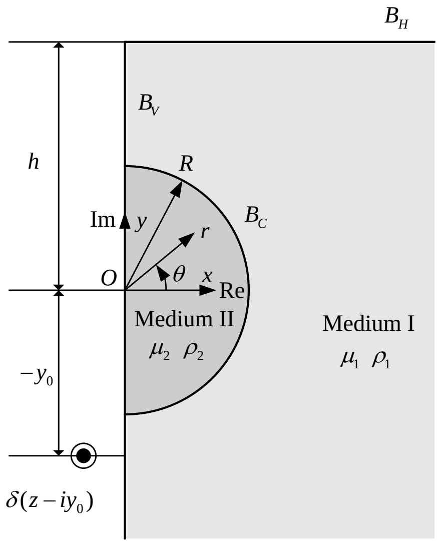

Figure1. Point source in the quarter space.

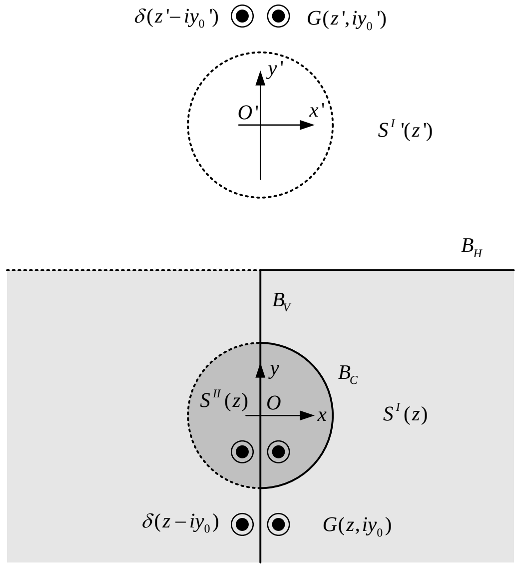

Figure2. Application of mirror image method .

In this way, the mass point in the elastic quarter space and semi-circular cylinder satisfy the wave equation (1) and (2) respectively.

$$\mu_1 \Delta w + f = \rho_1 \frac{\partial^2 w}{\partial t^2} \tag{1}$$

$$\mu_2 \Delta w + f = \rho_2 \frac{\partial^2 w}{\partial t^2} \tag{2}$$

In the formula, Δ is a two-dimensional Laplace operator, w is anti-plane displacement, f is anti-plane force. Considering the steady state problem, separate the time variable t and the space variable $(x,y),(r,\theta)$, obtain no homogeneous Helmholtz equation (3) and (4) of displacement.

$$\Delta w + k_1{}^2 w = -\frac{f^{\omega}}{\mu_1} \tag{3}$$

$$\Delta w + k_2{}^2 w = -\frac{f^{\omega}}{\mu_2} \tag{4}$$

$$k_1 = \frac{\omega}{c_1}, c_1 = \left(\frac{\mu_1}{\rho_1}\right)^{1/2} \tag{5}$$

$$k_2 = \frac{\omega}{c_2}, c_2 = \left(\frac{\mu_2}{\rho_2}\right)^{1/2} \tag{6}$$

In the formula, k_1 and k_2 are the numbers of shear waves in the quarter space and semi-circular cylinder respectively, c_1 and c_2 are the phase velocity of shear wave in the quarter space and semi-circular cylinder respectively, f^{ω} is out-of-plane loading which neglecting the time factor $\exp(-i\omega t)$. ω is angular frequency of steady state motion.

According to the description of functions (7), introducing complex variable z and conjugate complex variable $\overline{z}$, create a complex plane $(z,\overline{z})$, among them, $\mathrm{i}=\sqrt{-1}$ is imaginary number.

$$\begin{cases} z = x + iy = re^{i\theta} \\ \overline{z} = x - iy = re^{-i\theta} \end{cases} \tag{7}$$

There is anti-plane steady point source load $\delta(z-iy_0)\exp(-i\omega t)$ on the Point iy_0 of the vertical boundary plane B_V , which is substituted into the no homogeneous Helmholtz equation (3) and (4). That is $f^{\omega}=\delta(z-iy_0)$, and the solutions are (8) and (9).

$$G(z,iy_0)=\frac{i}{4\mu_1}H_0^{(1)}\left(k_1\left|z-iy_0\right|\right) \tag{8}$$

$$G(z,iy_0)=\frac{i}{4\mu_2}H_0^{(1)}\left(k_2\left|z-iy_0\right|\right) \tag{9}$$

They are the basic solutions of point source functions in the whole space. Among them, $\delta(\bullet)$ is the Dirac function, and $H_0^{(1)}(\bullet)$is the Hankel function of the first kind of 0 order.

In this paper, the governing equations solution of the mathematical physics equations of the anti-plane elastodynamics are the governing equations (3) and (4), the solution conditions are on the horizontal, vertical and semi-cylinder boundaries. They are respectively homogeneous Neumann conditions (10) and (11) on the plane B_H and plane B_V , and the homogeneous Dirichlet condition (12) and Neumann condition (13) on the circular cylinder B_C .

$$\tau_{yz}^{I}\Big|_{y=h}=\mu_1\frac{\partial w^I}{\partial y}\Big|_{y=h}=0 \tag{10}$$

$$\begin{cases} \tau_{xz}^{I}\Big|_{x=0}=\mu_1\dfrac{\partial w^I}{\partial x}\Big|_{x=0}=0 \\ \tau_{xz}^{II}\Big|_{x=0}=\mu_2\dfrac{\partial w^{II}}{\partial x}\Big|_{x=0}=0 \end{cases} \tag{11}$$

$$w^I\Big|_{r=R}=w^{II}\Big|_{r=R} \tag{12}$$

$$\tau_{rz}^{I}\Big|_{r=R}=\mu_1\frac{\partial w^I}{\partial r}\Big|_{r=R}=\tau_{rz}^{II}\Big|_{r=R}=\mu_2\frac{\partial w^{II}}{\partial r}\Big|_{r=R} \tag{13}$$

Here the components of stress and the superscript of displacements I and II are respectively represented in the quarter space Medium I and semi-circular cylinder Medium II.

The anti-plane displacement w and the space variable (x,y) in the plane of (O,x,y) are decoupled. According to the method of mirror image, the definite solution can be constructed. As shown in Figure 2, taking the plane B_V of the vertical boundary as the symmetry plane, extending the quarter space into half space, extending the semi-circular cylinder into cylinder, exerting symmetric anti plane steady point source loads $\delta(z-iy_0)\exp(-i\omega t)$ in the extended region. According to such extension processing, the exact solution of the problem is exactly equivalent, and satisfy the Neumann condition (11) on the plane B_V . This is steady state response of an elastic half space with a cylindrical inclusion on the surface of a symmetric plane. Here, the time harmonic function was omitted.

3. Point source load

The external force of the equivalent problem is the point source load $2\delta(z-iy_0)$. When it acts in the extended cylinder, that is $|y_0|<R$, the displacement field G^{II} is in accordance with equation(13), Its radial stress component τ_{rz} is denoted as G_{rz}^{II} . According to equation(14), $H_{\#}^{(1)}(\bullet)$ is #order the first kind of Hankel functions.

When $y_0\le h$ and $|y_0|\ge R$, the influence of horizontal boundary should be considered. As shown in figure 2, built a plane rectangular coordinate system (O',x',y') , then the complex variable z and the conjugate complex variable $\overline{z}$ can satisfy the equation (16). According to the method of mirror image, the displacement field G^I of the point load $2\delta(z-iy_0)$ is the equation (17), radial stress component τ_{rz} is denoted as G_{rz}^I, that is the equation(18).Where $y_0'=-y_0$, iy_0' is complex value of iy_0 that is the mirror point of the plane B_H on the complex plane.

$$G^{II}(z,iy_0)=\frac{i}{2\mu_2}H_0^{(1)}\left(k_2\left|z-iy_0\right|\right) \tag{14}$$

$$G_{rz}^{II}(z,iy_0)=\frac{ik_2}{8}\left[H_{-1}^{(1)}\left(k_2\left|z-iy_0\right|\right)-H_1^{(1)}\left(k_2\left|z-iy_0\right|\right)\right]\left[\left(\frac{z-iy_0}{\left|z-iy_0\right|}\right)^{-1}\left(\frac{z}{\bar{z}}\right)^{1/2}+\left(\frac{z-iy_0}{\left|z-iy_0\right|}\right)\left(\frac{\bar{z}}{z}\right)^{1/2}\right] \tag{15}$$

$$\begin{cases} z'=x'+iy'=z-2ih \\ \bar{z}'=x'-iy'=\bar{z}+2ih \end{cases} \tag{16}$$

$$G^I(z,iy_0)=\frac{i}{2\mu_1}H_0^{(1)}\left(k_1\left|z-iy_0\right|\right)+\frac{i}{2\mu_1}H_0^{(1)}\left(k_1\left|z'-iy_0'\right|\right) \tag{17}$$

$$\begin{aligned} G_{rz}^{I}(z,iy_0)=&\frac{ik_1}{8}\left[H_{-1}^{(1)}\left(k_1\left|z-iy_0\right|\right)-H_1^{(1)}\left(k_1\left|z-iy_0\right|\right)\right]\left[\left(\frac{z-iy_0}{\left|z-iy_0\right|}\right)^{-1}\left(\frac{z}{\bar{z}}\right)^{1/2}+\left(\frac{z-iy_0}{\left|z-iy_0\right|}\right)\left(\frac{\bar{z}}{z}\right)^{1/2}\right] \\ &+\frac{ik_1}{8}\left[H_{-1}^{(1)}\left(k_1\left|z'-iy_0'\right|\right)-H_1^{(1)}\left(k_1\left|z'-iy_0'\right|\right)\right]\left[\left(\frac{z'-iy_0'}{\left|z'-iy_0'\right|}\right)^{-1}\left(\frac{z}{\bar{z}}\right)^{1/2}+\left(\frac{z'-iy_0'}{\left|z'-iy_0'\right|}\right)\left(\frac{\bar{z}}{z}\right)^{1/2}\right] \end{aligned} \tag{18}$$

4. Displacement wave of cylinder

According to Fourier-Hankel wave function expansion, the extending of cylinder will engender the scattered wave in the half space. Its displacement field denoted as S^I can be given in the equation (19). A_n is the undetermined coefficient of wave function. Radial stress component τ_{rz} of scattering wave is denoted as S_{rz}^I, expressed as the series form in the equation (20), where $H_n^-(z)$ and $H_n^+(z)$ are the intermediate variable that can be obtained by differentiation, they can be defined as the equation (20).

$$S^I(z)=\sum_{n=-\infty}^{+\infty}A_nH_n^{(1)}\left(k_1\left|z\right|\right)\left(\frac{z}{\left|z\right|}\right)^n \tag{19}$$

$$S_{rz}^I(z)=\sum_{n=-\infty}^{+\infty}A_n\left[H_n^-(z)\left(\frac{z}{\bar{z}}\right)^{1/2}+H_n^+(z)\left(\frac{\bar{z}}{z}\right)^{1/2}\right] \tag{20}$$

$$\begin{cases} H_n^-(z)=\dfrac{k_1\mu_1}{2}H_{n-1}^{(1)}\left(k_1\left|z\right|\right)\left(\dfrac{\bar{z}}{z}\right)^{1/2}\left(\dfrac{z}{\left|z\right|}\right)^n \\ H_n^+(z)=-\dfrac{k_1\mu_1}{2}H_{n+1}^{(1)}\left(k_1\left|z\right|\right)\left(\dfrac{z}{\bar{z}}\right)^{1/2}\left(\dfrac{z}{\left|z\right|}\right)^n \end{cases} \tag{21}$$

By the method of mirror image, to built the mirror scattered wave that be engendered by the extending of cylinder which is the image of the plane B_H. The displacement field $S^I{}'$ is expressed as the series form in the equation (22), radial stress component τ_{rz} is denoted as $S_{rz}^I{}'$, expressed as the series form in the equation (23), where $H_{-n}^-(z')$ and $H_{-n}^+(z')$ are the intermediate variable that can be obtained by differentiation, and they can be defined as the equation (24).

$$S^I{}'(z)=\sum_{n=-\infty}^{+\infty}A_nH_n^{(1)}\left(k_1\left|z'\right|\right)\left(\frac{z'}{\left|z'\right|}\right)^{-n} \tag{22}$$

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$$S_{rz}^{I}{}'(z) = \sum_{n=-\infty}^{+\infty} A_n \left[H_{-n}^{-}(z')\left(\frac{z}{\bar{z}}\right)^{1/2} + H_{-n}^{+}(z')\left(\frac{\bar{z}}{z}\right)^{1/2} \right] \tag{23}$$

$$\begin{cases} H_{-n}^{-}(z') = -\dfrac{k_1\mu_1}{2} H_{n+1}^{(1)}\left(k_1\left|z'\right|\right)\left(\dfrac{\bar{z}'}{z'}\right)^{1/2}\left(\dfrac{z'}{\left|z'\right|}\right)^{-n} \\ H_{-n}^{+}(z') = \dfrac{k_1\mu_1}{2} H_{n-1}^{(1)}\left(k_1\left|z'\right|\right)\left(\dfrac{z'}{\bar{z}'}\right)^{1/2}\left(\dfrac{z'}{\left|z'\right|}\right)^{-n} \end{cases} \tag{24}$$

According to Fourier-Bessel wave function expansion, the extending of cylinder will engender the standing wave. Its displacement field denoted as S^{II} can be given in the series form in the equation (25). Where, B_n is the undetermined coefficient of wave function. $J_n(\bullet)$ is *n* order Bessel functions. Radial stress component τ_{rz} of standing wave is denoted as S_{rz}^{II}, expressed as the series form in the equation (26), where $J_n^{-}(z)$ and $J_n^{+}(z)$ are the intermediate variable that can be obtained by differentiation, they can be defined as the equation (27).

$$S^{II}(z) = \sum_{n=-\infty}^{+\infty} B_n J_n\left(k_2\left|z\right|\right)\left(\frac{z}{\left|z\right|}\right)^{n} \tag{25}$$

$$S_{rz}^{II}(z) = \sum_{n=-\infty}^{+\infty} B_n \left[J_n^{-}(z)\left(\frac{z}{\bar{z}}\right)^{1/2} + J_n^{+}(z)\left(\frac{\bar{z}}{z}\right)^{1/2} \right] \tag{26}$$

$$\begin{cases} J_n^{-}(z) = \dfrac{k_2\mu_2}{2} J_{n-1}\left(k_2\left|z\right|\right)\left(\dfrac{\bar{z}}{z}\right)^{1/2}\left(\dfrac{z}{\left|z\right|}\right)^{n} \\ J_n^{+}(z) = -\dfrac{k_2\mu_2}{2} J_{n+1}\left(k_2\left|z\right|\right)\left(\dfrac{z}{\bar{z}}\right)^{1/2}\left(\dfrac{z}{\left|z\right|}\right)^{n} \end{cases} \tag{27}$$

So, we can obtain the expressions of the equivalent displacement field w and radial stress component τ_{rz} in the elastic half-space and the cylinder.

5. Conclusion

According to the mirror method, extend the elastic quarter space to half space, and then extend it to a circular cylinder. By the wave function expansion method, the steady state of the displacement wave is constructed. It is generated the equivalent problem. The Fourier series expansion is performed on the continuity condition of the extended cylindrical boundary, then the coefficient of wave function is determined, and the steady solution of the problem is obtained. The radial stress and circumferential stress components are introduced, and the dynamic stress of the extended cylinder is analyzed quantitatively. It describes the anti plane scattering of the steady state point source by an interface semi-circular cylinder.

Results show (1), the scattering of the point source load on the interface semi-circular cylinder is clearly divided into the projection side and the shadow side. There is a clear boundary between the projection side and the shadow side of the radial stress, and the maximum value of the projection side and the shadow side is obtained on the line of the center point and the point source. With the increase of the shear modulus of the semi-circular cylinder, the maximum value of the radial stress is almost several times that of the shield side.(2) The circumferential stress also achieves the maximum value at the projection side, but it is not in the way of point source load, but there is a deflection angle.(3) In general, with the increase of the elastic modulus, the radial stress will increase and the circumferential stress will increase.(4) Similar to the scattering of plane SH waves by a cylindrical surface, the relatively low frequency point source load can always cause the large relative dynamic stress on the cylinder surface.

Reference

[1] Pao Y H and Mow C C 1973 *Diffraction of Elastic Waves and Dynamic Stress Concentrations* （New York: Crane and Russak）

[2] Liu D K and Liu H W 1998 Scattering of SH wave and dynamic stress concentration near the interface *Journal of Mechanics* **30**(5) 597-604

[3] Liu D K and Shi S X 2002 Scattering of plane SH waves by circular lining structure *Journal of mechanics* **34**(5) 796-803

[4] Liu D K and Tian Jia Yong 1999 Scattering and dynamic stress concentration of SH wave in cylindrical elastic inclusion *Explosion and Shock Waves* **19**(2) 115-23

[5] Tian Jia Yong, LIU Dian kui and LIU Chun Xu 1999 Far field solution of SH-wave scattered by interface cylindrical rigid inclusion *Journal of Harbin Engineering University* **20**(4) 90-7

[6] Qi Hui, Liu D K and Shi S X 2001 The displacement field of elastic half space with half circular lining *Journal of Harbin Engineering University* **22**(2) 81-5

[7] Qi Hui, Wang Yan and Liu D K 2003 Scattering of SH waves by circular waves in a semi infinite space interface earthquake engineering and engineering vibration *Journal of Earthquake Engineering and Engineering Vibration* **23**(3) 41-6

[8] Zhao Jia Xi, Qi Hui and Su S W 2008 An approximate analysis of the scattering of SH waves by a cylindrical elastic inclusion with an interfacial debonding *Applied Mathematics and Mechanics* **29**(6) 705-12

[9] Qiu Fa Qiang, Wang Hui Wen and Wang Xue 2010 The interaction between non isosceles triangular structure and foundation under SH wave incidence *Journal of Harbin Institute of Technology* **42**(4) 634-8

[10] Yang Z L, Xv H N and Chen Z G 2011 Scattering of SH waves by an isosceles triangular bulge and a semi-circular concave topography *Journal of Harbin Institute of Technology* S**1** 6-11

[11] Shi Yong, Qi Hui and Yang Zai Lin 2008 Scattering and ground motion of circular holes in the right plane region by SH wave *Chinese Journal of Applied Mechanics* **25**(3) 392-8

[12] Qi H, Shi Y and Liu D K 2009 Interaction of a circular cavity and a beeline crack in right-angle plane impacted by SH-wave *Journal of Harbin Institute of Technology* **16**(4) 548-53

[13] Qi Hui, Shi Yong, Li Hong Liang and Zhao Jia Xi 2009 Dynamic analysis of the circular hole near the interface between two phases in a vertical half space with SH wave incidence *Explosion and Shock Waves* **29**(1) 73-9

[14] Qi H, Yang J and Shi Y 2011 Scattering of SH-wave by cylindrical inclusion near interface in bi-material half space *Journal of Mechanics* **27**(1) 37-45

[15] Qi H, Yang J, Shi Y and Tian J Y 2012 Dynamic analysis for circular inclusion near interfacial crack impacted by SH wave in half space *Journal of Mechanics* **28**(1) 143-51

[16] Qi H and Yang J 2012 Dynamic analysis for circular inclusions of arbitrary positions near interfacial crack impacted by SH-wave in half-space *European Journal of Mechanics* A/Solids **36** 18-24