

A Thermoelastic Damping Model for the Cone Microcantilever Resonator with Circular Cross-section

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Abstract. Microbeams with variable cross-section have been applied in Microelectromechanical Systems (MEMS) resonators. Quality factor (Q -factor) is an important factor evaluating the performance of MEMS resonators, and high Q -factor stands for the excellent performance. Thermoelastic damping (TED), which has been verified as a fundamental energy lost mechanism for microresonators, determines the upper limit of Q -factor. TED can be calculated by the Zener's model and Lifshits and Roukes (LR) model. However, for microbeam resonators with variable cross-sections, these two models become invalid in some cases. In this work, we derived the TED model for cone microcantilever with circular cross-section that is a representative non-uniform microbeam. The comparison of results obtained by the present model and Finite Element Method (FEM) model proves that the present model is valid for predicting TED value for cone microcantilever with circular cross-section. The results suggest that the first-order natural frequencies and TED values of cone microcantilever are larger than those of uniform microbeam for large aspect ratios (l/r_0). In addition, the Debye peak value of a uniform microcantilever is equal to $0.5\Delta_E$, while that of cone microcantilever is about $0.438\Delta_E$.

1. Introduction

Micro-electro-mechanical System (MEMS) beam resonators with discontinuously or continuously variable cross-section are considered seriously due to their enhanced properties in real applications [1-2]. One of factors determining the performance of microbeam resonators is quality factor (Q -factor). A meaningful task for MEMS researchers and designers is to obtain a higher Q -factor of resonators. Thermoelastic damping (TED) is an intrinsic mechanism of energy lost in non-uniform microbeam resonators, which determines the upper limit of Q -factor. TED caused by an irreversible heat flux produced by temperature gradients cannot be completely avoided by the improved fabrication. Therefore, to accurately predict the TED value for the design and optimization of non-uniform microbeam resonators is of importance.

Zener [3-4], who firstly provided a 1D model of TED in the uniform cantilever, gave the TED formula of the beam with thickness h which is expressed as follow

$$Q_{\text{Zener}}^{-1} = \frac{1}{2\pi} \frac{\Delta W}{W} = \frac{E\alpha^2 T_0}{C_V} \frac{\omega\tau}{1+(\omega\tau)^2} \quad (1)$$

where ΔW represents the work lost per cycle, W represents the maximum stored energy per cycle. The thermal relax time $\tau = h^2 C_V / \pi^2 \kappa$, E is the elastic modulus (Young's modulus), T_0 denotes the equilibrium temperature, κ is the thermal conductivity, α denotes the coefficient of thermal expansion, C_V denotes the specific heat, and ω is the vibrational frequency.

Zener's model and LR's model are popularly used to calculate TED for uniform microbeam resonators.



Devices with circular cross-section have been produced and studied, such as nanowires [5], microposts [6]. In 2010, Tunvir et al studied the TED on hollow microresonators with circular cross-section [7]. In 2016, Li et al investigated the TED in microrings with circular cross-section [8]. However, Zener's model and LR's model are failed to provide an appropriate estimation in some cases [9]. Li and Hu [10], and Zhou et al [11] all pointed that LR's model and Zener's model are not valid to evaluate TED in non-uniform microbeam resonators. In this study, as a typical microbeam with continuous variable cross-section, TED model for the cone microcantilever with circular cross-section is derived based on Zener's theory.

2. TED definition of cone microcantilever

Figure 1 shows a simple uniform cantilever and a cone cantilever with circular cross-section. The cantilever is continuous, homogeneous and isotropic. The x -axis is along the microcantilever length direction, and the y -axis is along the microcantilever transverse vibrational direction. The r_0 and l represent right-end radius and length of microcantilever, respectively. The radius function $r(x)$ and area function $A(x)$ of cantilever shown in Figure 1.(b) are expressed as follows:

$$r(x) = r_0 \frac{x}{l}, \quad A(x) = \pi r_0^2 \left(\frac{x}{l} \right)^2 \quad (2)$$

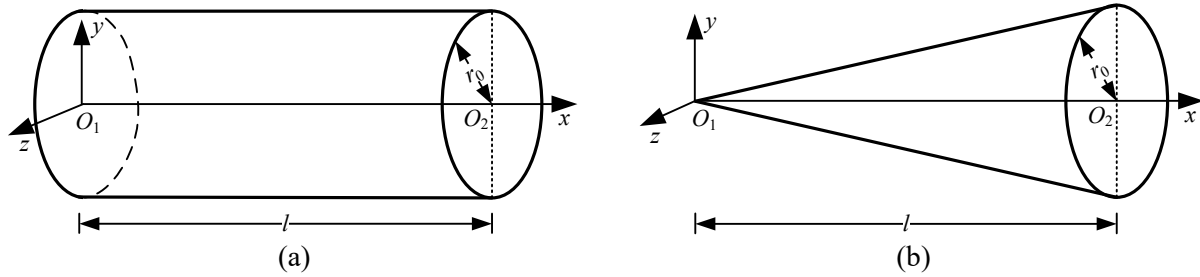


Figure 1. Schematic drawings of microcantilevers: (a) uniform microcantilever with circular cross-section, (b) cone microcantilever with circular cross-section.

Based on Euler-Bernoulli theory, the governing equation of free vibration in the x - y plane of cone microcantilever can be expressed as

$$\frac{\partial^2}{\partial x^2} [M_M(x) + M_T(x)] + \rho A(x) \frac{\partial^2 y(x,t)}{\partial t^2} = 0 \quad (3)$$

here ρ denotes the mass density, $y(x,t)$ represents the transverse vibration displacement, $M_T(x)$ represents the thermoelastic moment, and $M_M(x)$ represents the mechanical moment.

The governing equation of 2D heat conduction is based on the Principle of Fourier heat conduction, which can be expressed in the cylindrical coordinate system

$$\frac{\partial \theta}{\partial t} = \chi \left(\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \frac{\partial \theta}{\partial r} + \frac{\partial^2 \theta}{\partial \varphi^2} \right) + \frac{\Delta_E}{\alpha} \frac{\partial (y_{xx}(x,t))}{\partial t} r \sin \varphi \quad (4)$$

Where Δ_E is called Zener's modulus and $\Delta_E = E\alpha^2 T_0 / C_V$. By substituting $\theta(r, \varphi, t) = \hat{\theta}(r, \varphi) e^{i\omega t}$ and $y_{xx}(x, t) = Y_{xx}(x) e^{i\omega t}$ into above equation, we can obtain

$$i\omega \hat{\theta} = \chi \left(\frac{\partial^2 \hat{\theta}}{\partial r^2} + \frac{1}{r} \frac{\partial \hat{\theta}}{\partial r} + \frac{\partial^2 \hat{\theta}}{\partial \varphi^2} \right) + i\omega \cdot r \sin \varphi \cdot \frac{\Delta_E}{\alpha} Y_{xx} \quad (5)$$

In the paper [4], $\theta(x, y, t)$ is written as the total summation of thermal modes. It is worth noting that the first-order thermal mode affects thermoelastic energy dissipation seriously. Therefore, it is viable that $\theta(x, y, t)$ of cone microcantilever with circular cross-section can be written as the first-order thermal mode. Additionally, the thermal condition of the cone microcantilever surface is assumed to be adiabatic. At last, one can get the formula of $\theta_1(x, y, t)$

$$\hat{\theta}_1(x, r, \varphi) = \sin \varphi \frac{2r_0 \Delta_E}{\alpha} Y_{xx}(x) \frac{1}{(a_1^2 - 1) J_1(a_1)} \frac{i \omega \tau_1}{1 + i \omega \tau_1} J_1\left(\frac{a_1}{r_0} r\right) \quad (6)$$

here $\tau_1 = \frac{r^2}{\chi a_1^2}$, and $a_1 = 1.841$ [8].

The total energy dissipation over one period due to TED can be calculated by [8]

$$\Delta W = \pi \iiint_V \hat{\sigma} \operatorname{Im}(\hat{\varepsilon}_{\text{thermal}}) dV = 2\pi^2 E \Delta_E \cdot \frac{[2J_1(a_1) - a_1 J_0(a_1)]}{a_1^2 (a_1^2 - 1) J_1(a_1)} \int_0^l \frac{\omega \tau_1}{1 + (\omega \tau_1)^2} r^4(x) Y_{xx}^2(x) dx \quad (7)$$

here the amplitude of thermal strain in cone microcantilever $\hat{\varepsilon}_{\text{thermal}} = \alpha \hat{\theta}$, the amplitude of mechanical stress $\hat{\sigma} = E Y_{xx}(x) \cdot r \sin \varphi$, and Im represents the imaginary part.

The maximum energy stored W in the microcantilever can be calculated by [8]

$$W = \frac{1}{2} \iiint_V \hat{\sigma} \hat{\varepsilon}_{\text{elastic}} dV = \frac{E}{2} \int_0^l \int_0^{2\pi} \int_0^{r(x)} \sin^2 \varphi \cdot Y_{xx}^2(x) \cdot r^3 dr d\varphi dx = \frac{\pi E}{8} \int_0^l r^4(x) Y_{xx}^2(x) dx \quad (8)$$

According to the energy definition [12], TED model in the cone microcantilever with circular cross-section is given by

$$Q_{\text{TED}}^{-1} = \frac{1}{2\pi} \frac{\Delta W}{W} = \frac{8[2J_1(a_1) - a_1 J_0(a_1)]}{a_1^2 (a_1^2 - 1) J_1(a_1)} \cdot \Delta_E \cdot \frac{\int_0^l \frac{\omega \tau_1}{1 + (\omega \tau_1)^2} r^4(x) Y_{xx}^2(x) dx}{\int_0^l r^4(x) Y_{xx}^2(x) dx} \quad (9)$$

According to $\frac{8[2J_1(a_1) - a_1 J_0(a_1)]}{a_1^2 (a_1^2 - 1) J_1(a_1)} = 0.9877 \approx 1$ with respect to $a_1 = 1.841$, the equation (9) reduces to

$$Q_{\text{TED}}^{-1} = \Delta_E \cdot \frac{\int_0^l \frac{\omega \tau_1}{1 + (\omega \tau_1)^2} r^4(x) Y_{xx}^2(x) dx}{\int_0^l r^4(x) Y_{xx}^2(x) dx} \quad (10)$$

Then, the dimensionless form of equation (10) is written as follows

$$Q_{\text{TED}}^{-1} = \Delta_E \cdot \frac{\int_0^1 \frac{\eta \cdot X^6}{1 + \eta^2 X^4} Y_{XX}^2(X) dX}{\int_0^1 X^4 Y_{XX}^2(X) dX} \quad (11)$$

Where $\eta = \frac{\omega r_0^2}{\chi a_1^2}$. It is obvious that $Y(X)$ is an important step to calculate thermoelastic damping, and the formula of $Y(X)$ will be derived in Section 3.

3. Modal parameters

3.1. Modal shape function

The non-dimensional radius function, non-dimensional area function, and non-dimensional inertia moment function are written as

$$r(X) = r_0 X, \quad A(X) = A_0 X^2, \quad I(X) = I_0 X^4 \quad (12)$$

here $X = x/l$, $A_0 = \pi r_0^2$, and $I_0 = \pi r_0^4/4$. Because $M_T(x)$ is neglected compared to $M_M(x)$, substitute $y(x, t) = Y(x) e^{i\omega t}$ into equation (3), then one can obtain the non-dimensional form of equation (3):

$$\frac{d^4 Y(X)}{dX^4} + \frac{8}{X} \frac{d^3 Y(X)}{dX^3} + \frac{12}{X^2} \frac{d^2 Y(X)}{dX^2} - \frac{\lambda^2}{X^2} Y(X) = 0 \quad (13)$$

where $\lambda^2 = \omega^2 \rho A_0 l^4 / EI_0$ and ω is the natural angular frequency of the cantilever.

The solution of equation (13) is given by [13]

$$Y(X) = \frac{1}{X} \left[C_1 J_2(2\sqrt{\lambda X}) - C_2 I_2(2\sqrt{\lambda X}) \right] \quad (14)$$

here J_2 is the 2nd order Bessel function of the first kind with real argument, and I_2 is the 2nd order Bessel function of the first kind with imaginary argument. The C_1 and C_2 are undetermined coefficients.

3.2. Modal eigen frequency

Considering boundary conditions of the cone microcantilever, the rotation and displacement of the right-end are zero, $Y(1) = Y'(1) = 0$, then the following equations are

$$\begin{cases} J_2(2\sqrt{\lambda})C_1 - I_2(2\sqrt{\lambda})C_2 = 0 \\ \left[\sqrt{\lambda}J_1(2\sqrt{\lambda}) - 2J_2(2\sqrt{\lambda}) \right]C_1 + \left[-\sqrt{\lambda}I_1(2\sqrt{\lambda}) + 2I_2(2\sqrt{\lambda}) \right]C_2 = 0 \end{cases} \quad (15)$$

For meaningful solutions of C_1 and C_2 , the eigen frequency equation is computed by

$$\begin{vmatrix} J_2(2\sqrt{\lambda}) & -I_2(2\sqrt{\lambda}) \\ \sqrt{\lambda}J_1(2\sqrt{\lambda}) - 2J_2(2\sqrt{\lambda}) & -\sqrt{\lambda}I_1(2\sqrt{\lambda}) + 2I_2(2\sqrt{\lambda}) \end{vmatrix} = 0 \quad (16)$$

And equation (16) can be expanded to be

$$J_1(2\sqrt{\lambda})I_2(2\sqrt{\lambda}) - J_2(2\sqrt{\lambda})I_1(2\sqrt{\lambda}) = 0 \quad (17)$$

Solve equation (17) in MATLAB and obtain the values of λ . By setting $C_2 = 1$ and $C_1 = \gamma = I_2(2\sqrt{\lambda})/J_2(2\sqrt{\lambda})$, the final the square of $Y_{xx}(X)$ is obtained by

$$Y_{xx}^2(X) = \frac{1}{X^7} \left[\lambda X^{1.5} (\gamma J_0 - I_0) + 4\sqrt{\lambda}X (-\gamma J_1 + I_1) - 6\sqrt{X} (\gamma J_0 + I_0) + \frac{6}{\sqrt{\lambda}} (\gamma J_1 + I_1) \right]^2 \quad (18)$$

where J_0, J_1, I_0 and I_1 are all Bessel functions of the first kind with respect to $2\sqrt{\lambda X}$.

By substituting equation (18) and structural geometry functions equation (12) into equation (8), then the corresponding value of TED can be computed by integration. Here we notice that

$$\omega = \lambda \sqrt{\frac{EI_0}{\rho A_0 I^4}} \quad (19)$$

and $\lambda = 8.7193, 21.1457, 38.4538, \dots$

4. Validation and discussion

In this section, the results computed by the present model for cone microcantilevers with circular cross-section are compared with the FEM results obtained by ANSYS. Solid226, which is a 3D 20-Node element, is applied in coupled thermal-structural analysis. A harmonic force $p \sin(\omega t)$ excites the cone microcantilever. The single-crystal silicon is applied in simulation. And the material properties of silicon [14] are listed as follows: $E = 160 \text{ GPa}$, $\rho = 2330 \text{ Kg/m}^3$, $C_V = 1.6 \times 10^6 \text{ J/m}^3/\text{K}$, $\kappa = 120 \text{ W/m/K}$, $\alpha = 2.6 \times 10^{-6} \text{ K}^{-1}$, $\chi = 7.4033 \times 10^{-5} \text{ m}^2/\text{s}$, $\nu = 0.22$, and $T_0 = 300 \text{ K}$. In real applications, only TED at fundamental frequencies is of interest [7, 11].

Firstly, according to equation (19), the first-order natural frequencies of cone microcantilever are calculated. Figures 2 and 3 show the fundamental frequency against the beam length with the constant r_0 is equal to $2.5 \mu\text{m}$ and $5 \mu\text{m}$, respectively. It is worth noting that for the same r_0 , the frequency of cone microcantilever is larger than that of uniform microcantilever. Therefore, when the resonators need to work at high frequencies, it is better to choose cone cantilever instead of uniform cantilever.

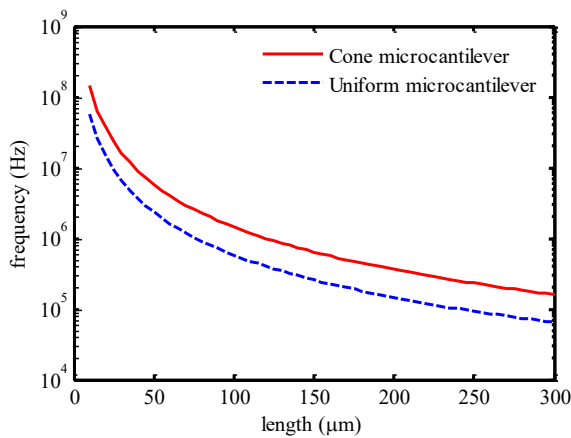


Figure 2. Comparison of first-order natural frequencies against the length. The radius r_0 of cone cantilever and uniform cantilever is $2.5\mu\text{m}$.

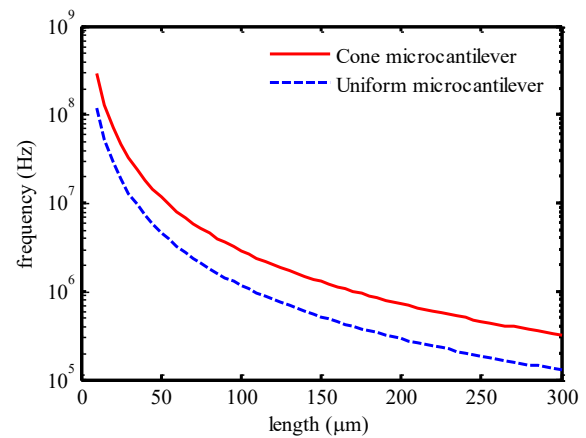


Figure 3. Comparison of first-order natural frequencies against the length. The radius r_0 of cone cantilever and uniform cantilever is $5\mu\text{m}$.

Comparison of TED values obtained from the present model and FEM model are plotted in Figures 4 and 5 for the representative radius r_0 . As expected, results based on the present model agree well with those based on the FEM model, and the relative errors in two cases are less than 3% over the wide range of beam length. As observed from Figure 5, TED values of cone microcantilever are larger than those of uniform microbeam with increasing aspect ratios (l/r_0). Figures 4 and 5 display the perfect agreement proving that the present model is correct and effective.

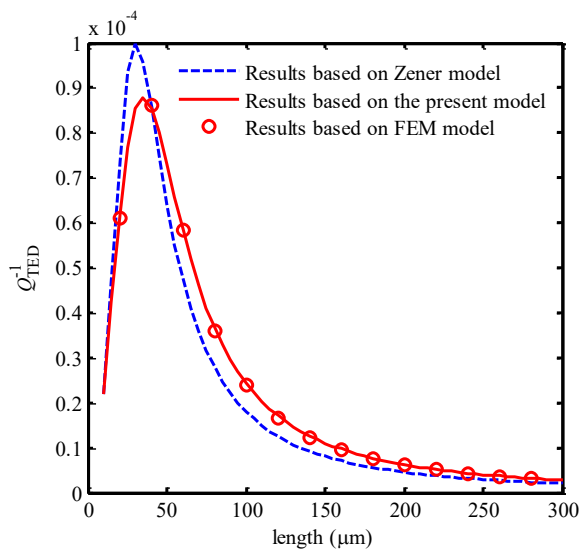


Figure 4. Comparison of TED values against the beam length. The radius r_0 of uniform beam and cone cantilever are both $2.5\mu\text{m}$.

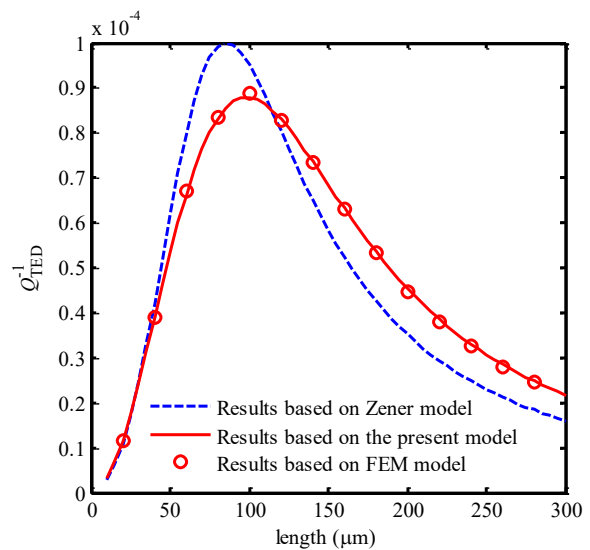


Figure 5. Comparison of TED values against the beam length. The radius r_0 of uniform beam and cone cantilever are both $5\mu\text{m}$.

On the other hand, Figure 6 shows that the Debye peak value of cone microcantilever with circular cross-section is about $0.438\Delta_E$ at $\eta=2$, while that of Zener's model is $0.5\Delta_E$ at $\eta=1$. Moreover, the results suggest that for non-uniform microbeam resonators, TED is dependent of both the vibrational frequencies and the mode shape functions.

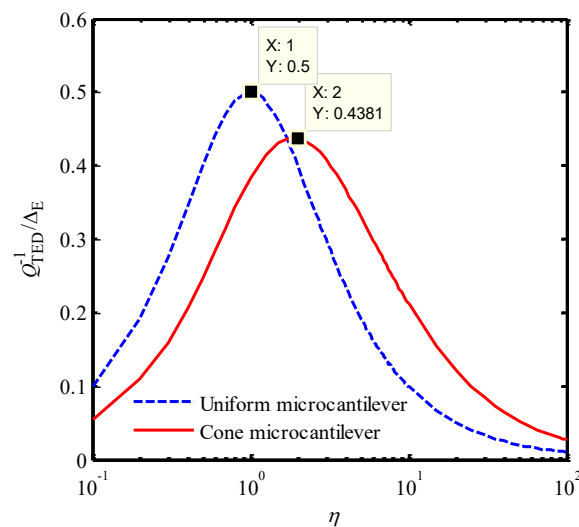


Figure 6. Variation of normalized TED curves for uniform microcantilever and cone microcantilever.

5. Conclusions

In this work, TED model for the cone microcantilever resonator with circular cross-section are presented and analyzed. Compared with numerical results obtained from the FEM model, and perfect agreements indicate that the proposed TED model is correct and reasonable. The current work could help MEMS designers to achieve better performance (higher Q -factor) by reducing the effect of thermoelastic damping.

Acknowledgements

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