

Thermoelastic Damping in Cone Microcantilever Resonator

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Abstract. Microbeams with continuous or discontinuous variable cross-section have been applied in Microelectromechanical Systems (MEMS) resonators, such as tapered microbeam, torsion microbeam and stepped microbeam. Thermoelastic damping (TED), which is verified as a fundamental energy lost mechanism for microresonators, is calculated by the Zener's model and Lifshits and Roukes's (LR) model in general. However, for non-uniform microbeam resonators, these two classical models are not suitable in some cases. On the basis of Zener's theory, a TED model for cone microcantilever with rectangular cross-section has been derived in this study. The comparison of results obtained by the present model and Finite Element Method (FEM) model proves that the proposed model is able to predict TED value for cone microcantilever. In addition, TED in cone microcantilever is nearly same as TED in wedge microcantilever. The results show that quality factors (Q -factors) of cone microcantilever and wedge microcantilever are larger than Q -factor of uniform microbeam at low frequencies. The Debye peak value of a uniform microcantilever is equal to $0.5\Delta_E$, while those of cone microcantilever and wedge microcantilever are about $0.438\Delta_E$ and $0.428\Delta_E$, respectively.

1. Introduction

With a fast growth of advanced manufacture technologies of Microelectromechanical Systems (MEMS), microbeam resonators with discontinuously or continuously variable cross-section have already been applied in real applications [1-3]. Non-uniform microbeam resonators are considered seriously due to their enhanced properties. A permanent topic for MEMS researchers and designers is to obtain a higher quality factor (Q -factor). Thermoelastic damping (TED) is a significant and fundamental mechanism of energy lost in non-uniform microbeam resonators. Especially, TED determines the Q -factor of microresonators. TED is caused by an irreversible heat flux produced by temperature gradients in the vibrating microbeam. It is unfortunate that TED cannot be entirely avoided by the improved design and fabrication. Hence, to accurately predict the TED value for the design and optimization of non-uniform microbeam resonators is of importance. Zener [4-5] firstly provided a 1D model of TED in a uniform reed (beam) in 1937 and 1938. And the TED formula of Zener's model for the beam with thickness h is expressed as follow

$$Q_{\text{Zener}}^{-1} = \frac{1}{2\pi} \frac{\Delta W}{W} = \frac{E\alpha^2 T_0}{C_V} \frac{\omega\tau}{1+(\omega\tau)^2} \quad (1)$$

where ΔW represents the work lost per cycle, W represents the maximum stored energy per cycle. The thermal relax time $\tau = h^2 C_V / \pi^2 \kappa$, E is the elastic modulus (Young's modulus), T_0 denotes the equilibrium temperature, κ is the thermal conductivity, α denotes the coefficient of thermal expansion, C_V denotes the specific heat, and ω is the vibrational frequency. Lifshitz and Roukes (LR) [6] proposed an accurate formula for TED in the uniform beam in 2000, which is given by



$$Q_{LR}^{-1} = \frac{E\alpha^2 T_0}{C_V} \frac{6}{\xi^2} \left[1 - \frac{1}{\xi} \left(\frac{\sinh \xi + \sin \xi}{\cos \xi + \cosh \xi} \right) \right] \quad (2)$$

here the dimensionless variable $\xi = h \sqrt{\frac{\omega C_V}{2\kappa}}$.

Zener's model and LR's model are widely used to calculate TED for uniform microbeam resonators. However, Zener's model and LR's model are failed to provide an appropriate estimation in some cases. In 2006, Candler et al [7] studied TED in the slotted microbeam. They compared their experimental Q -factors with the theoretical Q -factors obtained from Zener's model, but found that Zener's model is not able to give an appropriate estimation. In 2011, Li et al [8] showed that LR's model is effective for microresonators with added proof masses when all supported cantilevers have the same and constant thickness. In 2013, Tunvir [9] investigated TED in double fixed stepped microbeams, and studied the effect of step position on thermoelastic dissipation. In 2016, Zhou et al [10] investigated TED in the wedge microcantilever resonator with rectangular cross-section. As papers reported above, Li et al [8] and Tunvir [9] both suggested that LR's model and Zener's model are not valid to evaluate TED in non-uniform microbeam resonators. The cone microcantilever is another typical microbeam with discontinuous variable cross-section, so it is meaningful to develop the corresponding TED model. In this study, TED model for the cone microcantilever whose width and thickness changes linearly is derived on the basis of Zener's theory [4].

2. TED definition of cone microcantilever

Figure 1 shows three types of microcantilevers with rectangular cross-section. Figure 1(a) shows a simple uniform cantilever, Figure 1(b) shows a wedge cantilever, and Figure 1(c) shows a cone cantilever. The cantilever is continuous, homogeneous and isotropic. The x -axis is along the microcantilever length direction, and the y -axis is along the microcantilever thickness direction. The h , b and l represent right-end thickness, width, and length of microcantilever, respectively. And the thickness function $h(x)$ and cross-sectional area function $A(x)$ of the cone microcantilever shown in Figure 1(c) are $h(x) = hx/l$ and $A(x) = bh(x)$, respectively.

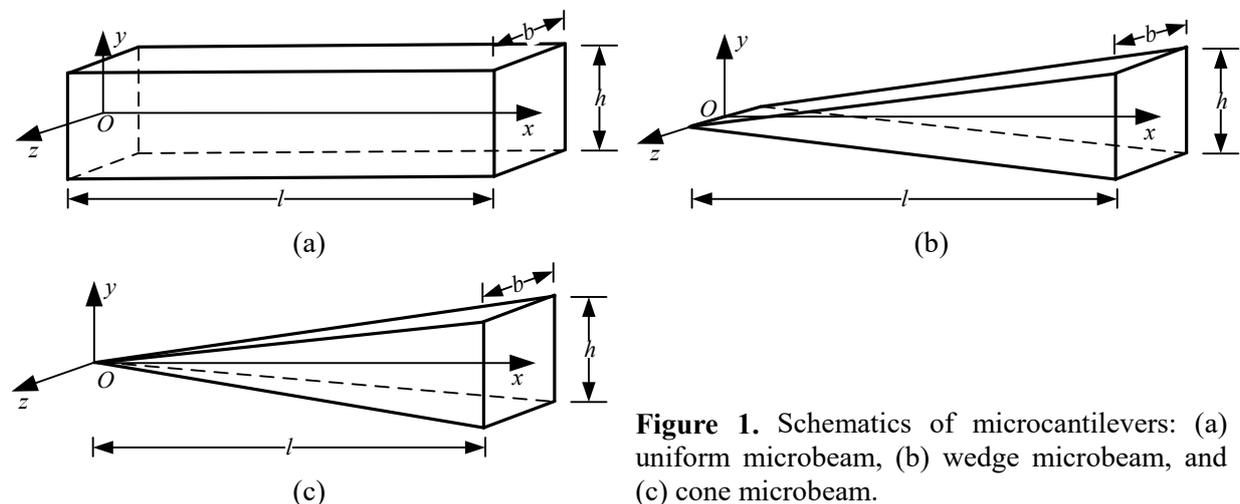


Figure 1. Schematics of microcantilevers: (a) uniform microbeam, (b) wedge microbeam, and (c) cone microbeam.

Based on Euler-Bernoulli theory, the governing equation of free vibration in the x - y plane of cone microcantilever can be expressed as

$$\frac{\partial^2}{\partial x^2} [M_M(x) + M_T(x)] + \rho A(x) \frac{\partial^2 y(x,t)}{\partial t^2} = 0 \quad (3)$$

here ρ denotes the mass density, $y(x,t)$ represents the transverse vibrational displacement, $M_T(x)$ represents the thermoelastic moment, and $M_M(x)$ represents the mechanical moment.

The governing equation of 1D (y -direction) heat conduction is based on the Principle of Fourier heat conduction, which can be expressed as

$$\frac{\partial \theta(x, y, t)}{\partial t} = \chi \frac{\partial^2 \theta(x, y, t)}{\partial y^2} + y \frac{\Delta_E}{\alpha} \frac{\partial}{\partial t} \frac{\partial^2 y(x, t)}{\partial x^2} \quad (4)$$

here $\theta(x, y, t)$ represents the temperature field function, Δ_E is called Zener's modulus and $\Delta_E = E\alpha^2 T_0 / C_V$.

In the paper [4], $\theta(x, y, t)$ is written as the total summation of thermal modes. It is worth noting that the first-order thermal mode affects thermoelastic energy dissipation seriously. Therefore, it is viable that $\theta(x, y, t)$ of cone microcantilever can be written as the first-order thermal mode. Additionally, the thermal condition of the cone microcantilever surface is assumed to be adiabatic. At last, one can get the formula of $\theta_0(x, y, t)$

$$\theta_0(x, y, t) = \hat{\theta}_0(x, y) e^{i\omega t} \quad (5)$$

here $\hat{\theta}_0$ is the amplitude of θ_0 and written as

$$\begin{cases} \hat{\theta}_0(x, y) = \frac{E\alpha T_0}{C_V} Y_{xx}(x) \frac{i\omega}{i\omega + \mu_0} S_0 \sin\left(\frac{1}{h(x)} \pi y\right) \\ \mu_0(x) = \frac{\chi \pi^2}{h^2(x)} \\ S_0(x, y) = \frac{2}{h(x)} \int_{-h(x)/2}^{h(x)/2} y \sin\left(\frac{1}{h(x)} \pi y\right) dy = \frac{4h(x)}{\pi^2} \end{cases} \quad (6)$$

here $y(x, t) = Y(x) e^{i\omega t}$.

The total energy dissipation over one period due to TED can be calculated by [9]

$$\Delta W = \pi \iiint_V \hat{\sigma} \text{Im}(\hat{\epsilon}_{\text{thermal}}) dV = \pi \frac{bE^2 \alpha^2 T_0}{12C_V} \int_L \frac{\omega \chi \pi^2 h^5(x)}{\omega^2 h^4(x) + \chi^2 \pi^4} Y_{xx}^2 dx \quad (7)$$

here the amplitude of thermal strain in cone microcantilever $\hat{\epsilon}_{\text{thermal}} = \alpha \hat{\theta}$, the amplitude of mechanical stress $\hat{\sigma} = E y Y_{xx}$, and Im represents the imaginary part.

The maximum energy stored W in the microcantilever can be calculated by [9]

$$W = \frac{1}{2} \iiint_V \hat{\sigma} \hat{\epsilon}_{\text{elastic}} dV = \frac{bE}{24} \int_L h^3(x) Y_{xx}^2 dx \quad (8)$$

According to the energy definition [11], TED model in the cone microcantilever is given by

$$Q_{\text{TED}}^{-1} = \frac{1}{2\pi} \frac{\Delta W}{W} = \chi \pi^2 \Delta_E \frac{\int_L \frac{\omega b(x) h^5(x)}{\omega^2 h^4(x) + \chi^2 \pi^4} Y_{xx}^2 dx}{\int_L b(x) h^3(x) Y_{xx}^2 dx} \quad (9)$$

Without losing generality, the non-dimensional form of equation (9) can be

$$Q_{\text{TED}}^{-1} = \Delta_E \frac{\int_0^1 \frac{\eta X^6}{\eta^2 X^4 + 1} Y_{XX}^2(X) dX}{\int_0^1 X^4 Y_{XX}^2(X) dX} \quad (10)$$

where $\eta = \frac{\omega h^2}{\chi \pi^2}$. It is obvious that $Y(X)$ is an important step to calculate thermoelastic damping, and the formula of $Y(X)$ will be derived in Section 3.

3. Modal parameters

3.1. Modal shape function

The non-dimensional thickness function, non-dimensional width function, and non-dimensional cross-sectional area function are $h(X) = hX$, $b(X) = bX$, and $A(X) = b(X)h(X)$, respectively. Here, the non-dimensional variable $X = x/l$. Because $M_T(x)$ is neglected compared to $M_M(x)$, the non-dimensional form of equation (3) is simplified as following:

$$\frac{d^4 Y(X)}{dX^4} + \frac{8}{X} \frac{d^3 Y(X)}{dX^3} + \frac{12}{X^2} \frac{d^2 Y(X)}{dX^2} - \frac{\lambda^2}{X^2} Y(X) = 0 \quad (11)$$

where $A_0 = bh$, $I_0 = bh^3/12$, $\lambda^2 = \omega^2 \rho A_0 l^4 / EI_0$.

The solution of equation (11) is given by [12]

$$Y(X) = \frac{1}{X} \left[C_1 J_2(2\sqrt{\lambda X}) - C_2 I_2(2\sqrt{\lambda X}) \right] \quad (12)$$

here J_2 is the 2nd order Bessel function of the first kind with real argument, and I_2 is the 2nd order Bessel function of the first kind with imaginary argument. The C_1 and C_2 are undetermined coefficients.

3.2. Modal eigen frequency

Considering boundary conditions of the cone microcantilever, the rotation and displacement of the right-end are zero, $Y(1) = Y'(1) = 0$, then the following equations are

$$\begin{cases} J_2(2\sqrt{\lambda})C_1 - I_2(2\sqrt{\lambda})C_2 = 0 \\ \left[\sqrt{\lambda}J_1(2\sqrt{\lambda}) - 2J_2(2\sqrt{\lambda}) \right]C_1 + \left[-\sqrt{\lambda}I_1(2\sqrt{\lambda}) + 2I_2(2\sqrt{\lambda}) \right]C_2 = 0 \end{cases} \quad (13)$$

For meaningful solutions of C_1 and C_2 , the eigen frequency equation is given by

$$\begin{vmatrix} J_2(2\sqrt{\lambda}) & -I_2(2\sqrt{\lambda}) \\ \sqrt{\lambda}J_1(2\sqrt{\lambda}) - 2J_2(2\sqrt{\lambda}) & -\sqrt{\lambda}I_1(2\sqrt{\lambda}) + 2I_2(2\sqrt{\lambda}) \end{vmatrix} = 0 \quad (14)$$

And the equation (14) can be expanded to

$$J_1(2\sqrt{\lambda})I_2(2\sqrt{\lambda}) - J_2(2\sqrt{\lambda})I_1(2\sqrt{\lambda}) = 0 \quad (15)$$

Solve the equation (15) in MATLAB and obtain the λ . By setting $C_2 = 1$ and $C_1 = \gamma = I_2(2\sqrt{\lambda})/J_2(2\sqrt{\lambda})$, then the square of $Y_{xx}(X)$ is obtained by

$$Y_{xx}^2(X) = \frac{1}{X^7} \left[\lambda X^{1.5} (\gamma J_0 - I_0) + 4\sqrt{\lambda} X (-\gamma J_1 + I_1) - 6\sqrt{X} (\gamma J_0 + I_0) + \frac{6}{\sqrt{\lambda}} (\gamma J_1 + I_1) \right]^2 \quad (16)$$

J_0, J_1, I_0 and I_1 are all Bessel functions of the first kind with respect to $2\sqrt{\lambda X}$.

By substituting equation (16) and different structural geometry parameters into equation (10), then the corresponding value of TED can be computed by integration.

4. Validation and discussion

In this part, the results computed by the present model for cone microcantilevers are compared with the FEM results obtained by ANSYS. Solid226, which is a 3D 20-Node element, is applied in coupled thermal-structural analysis. A harmonic force $p \sin(\omega t)$ excites the cone microcantilever. The single-crystal silicon is applied in simulation. And the material properties of silicon [13] are listed in Table 1.

Firstly, the values of TED in first-ten order vibrational modes of wedge microcantilever [10] and cone microcantilever are investigated. For instance, these two kinds of microcantilevers have the same dimensions: $l = 300 \mu\text{m}$ and $h = b = 10 \mu\text{m}$. TED values against the modal order are shown in Figure 2.

As observed in Figure 2, results based on the present model match well with those based on the FEM model. The relative errors in two cases are smaller than 5% over the range of modal order. In reality, most of microbeam resonators only operate at their fundamental frequencies. TED of wedge microcantilever [10] and cone microcantilever in the first flexural mode as a function of vibrational frequencies are calculated as shown in Figure 3. As expected, the differences between the results

obtained from the current model and the FEM model are also no more than 5%, and can be neglected. Figure 3 and Figure 4 show the perfect agreement proving that the present model is effective.

Table 1. Values of material properties of single-crystal silicon

| Symbols | Units | Values | Symbols | Units | Values |
|---------|---------------------|-------------------|----------|-------------------|-------------------------|
| E | GPa | 160 | κ | W/m/K | 120 |
| ρ | Kg/m ³ | 2330 | α | K ⁻¹ | 2.6×10^{-6} |
| ν | / | 0.22 | χ | m ² /s | 7.4033×10^{-5} |
| C_V | J/m ³ /K | 1.6×10^6 | T_0 | K | 300 |

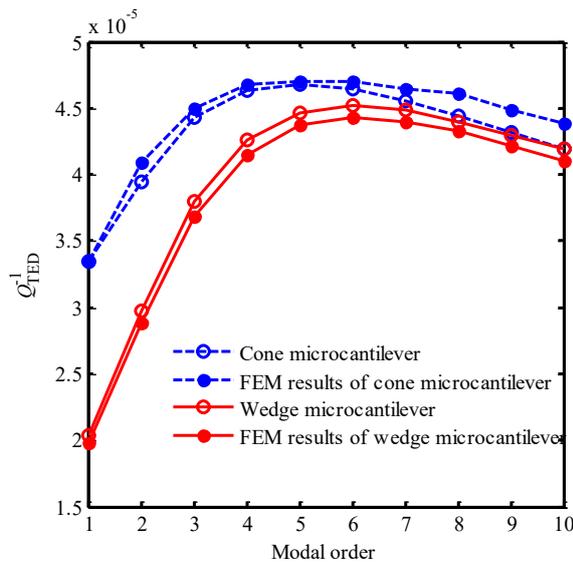


Figure 2. Comparison of TED values in first-ten order modes

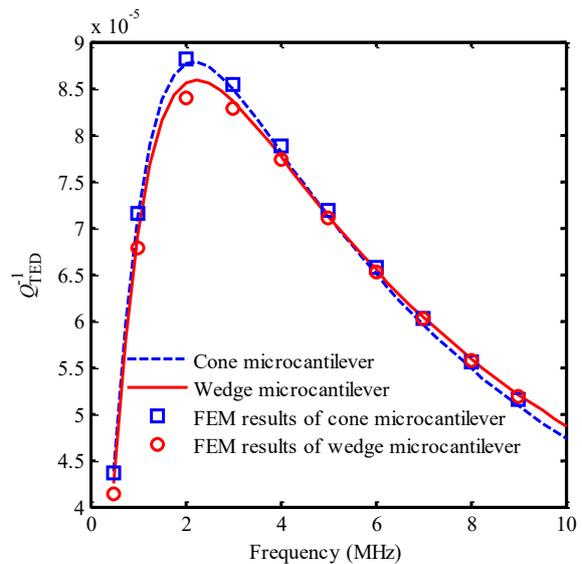


Figure 3. Comparison of TED values against frequencies. The two types of microcantilever resonators have the same dimension.

Figure 4 shows the normalized TED (Q_{TED}^{-1}/Δ_E) as a function of the variable η . There is almost no difference between TED curves of wedge microcantilever [10] and cone microcantilever. Moreover, this point suggests that TED in cone microcantilever does not depend on the width b , which is the same as the characteristic of wedge microcantilever [10]. However, the performance of wedge microcantilever and cone microcantilever is different from that the uniform microbeam. The small value of η stands for a low frequency but a large aspect ratio l/h . For η less than 1, TED values of the uniform microbeam are larger than those of two non-uniform microcantilevers, which means that two types of non-uniform microcantilevers own better performances and higher Q -factors. Thereby, for longer and slender microresonators, the cone or wedge microcantilever should be chose in real application instead of the uniform microcantilever. On the other hand, it is found that the Debye peak value of cone microcantilever is about $0.438\Delta_E$, while that of Zener's model is $0.5\Delta_E$. In the paper [10], the Debye peak value of wedge microcantilever is about $0.428\Delta_E$. Moreover, the results suggest that for non-uniform microbeam resonators, TED is dependent of frequencies and the mode shape.

Figure 5 shows the imaginary region of temperature profile in a typical cone microcantilever. One can observe that tensional regions of the cone microcantilever lose temperature while compressed regions generate temperature. This phenomenon of interest is right caused by TED.

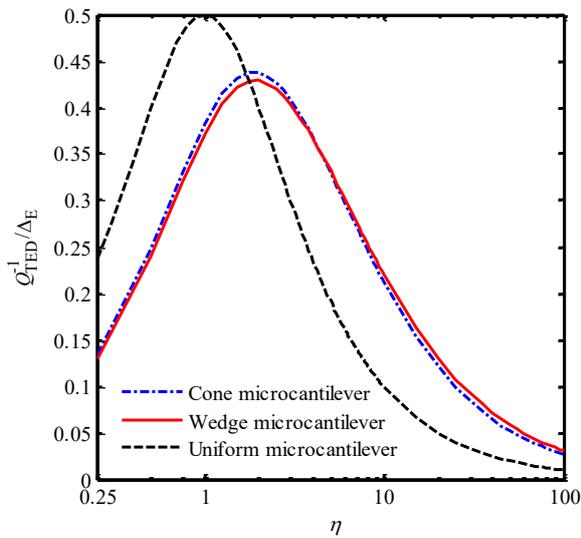


Figure 4. Comparison of normalized TED curves for three types of microcantilever resonators

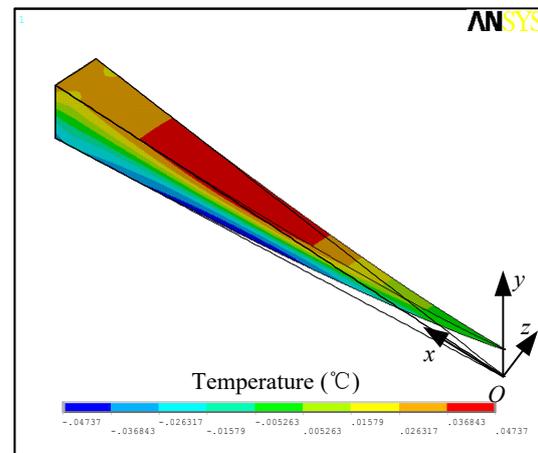


Figure 5. Imaginary region of a representative cone microcantilever

5. Conclusions

In this study, TED model for cone microcantilevers are presented and analyzed. Compared with numerical results obtained from the FEM model, and perfect agreements imply that the proposed TED model is correct and reasonable. The current work could help MEMS designers to achieve better performance (higher Q -factor) by reducing thermoelastic damping.

Acknowledgements

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