A comparison of the two approaches of the theory of critical distances based on linear-elastic and elasto-plastic analyses

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Abstract. The problem of determining the strength of engineering structures, considering the effects of the non-local fracture in the area of stress concentrators is a great scientific and industrial interest. This work is aimed on modification of the classical theory of critical distance that is known as a method of failure prediction based on linear-elastic analysis in case of elasto-plastic material behaviour to improve the accuracy of estimation of lifetime of notched components. Accounting plasticity has been implemented with the use of the Simplified Johnson-Cook model. Mechanical tests were carried out using a 300 kN electromechanical testing machine Shimadzu AG-X Plus. The cylindrical un-notched specimens and specimens with stress concentrators of titanium alloy Grade2 were tested under tensile loading with different grippers travel speed, which ensured several orders of strain rate. The results of elasto-plastic analyses of stress distributions near a wide variety of notches are presented. The results showed that the use of the modification of the TCD based on elasto-plastic analysis gives us estimates falling within an error interval of ±5-10%, that more accurate predictions than the linear elastic TCD solution. The use of an improved description of the stress-strain state at the notch tip allows introducing the critical distances as a material parameter.

1. Introduction
Nowadays the problem of determining the strength of engineering structures, considering the effects of the non-local fracture in the area of stress concentrators becomes especially acute due to the fact that this problem of fracture of materials under different loading conditions deals with a large number of applications in different engineering fields. This problem is significant in all the cases where geometrical features in structures give rise to localized stress concentration which may generate a crack leading to failure or to a shortening of the assessed lifetime of structures.

Today the method of failure prediction in cases where stress concentrations are present, known as the Theory of Critical Distances (TCD) [1], is one of the most popular methods. The fundamental idea on which the Theory of Critical Distances is based was first proposed in 1958 by Neuber [2] assumed that fatigue failure occurs when the mean stress with some distance from the notch root is equal to the fatigue strength of smooth specimen. One year later (1959), Peterson assumed that the fatigue failure occurs when the stress at one point which has a critical distance from the notch root is equal to the

fatigue strength of smooth specimen [3]. In 1969 Novozhilov introduced a necessary and sufficient criterion for estimating the strength of an elastic body weakened by a crack in the form of the average stress limitation in the cohesive zone length d ahead of the crack tip [4]. The above-mentioned ideas were the basis of four formalized method of the TCD such as the Point, the Line, the Area, and the Volume Method [5]. These four methods use one characteristic material length parameter, so-called critical distance L, to predict both brittle fracture and fatigue strength. The TCD takes a starting point the assumption that that the strength of notched components can be estimated by directly post-processing the entire linear-elastic stress field acting on the material in the vicinity of the stress concentrator [6,7]. It has also been proved that the TCD can be used to predict static fracture in ductile metallic materials containing various stress raisers and subjected to both uniaxial and multiaxial loading [8]. Moreover, in papers [9, 10] was shown that Theory of Critical Distances is suitable for predicting the strength of notched metallic materials subjected to dynamic loading.

In this study considered the modification of the classical theory of critical distance in cases of elasto-plastic material behavior for improve the accuracy of estimation of lifetime of notched components, because the material behavior being, by nature, highly non-linear and cannot be described in the framework of the linear theory of elasticity.

The paper is organized as follows. Section 2 is devoted to Theory of Critical Distances based on the linear-elastic analysis. There is considered the fundamentals of the theory of critical distances. The materials and experimental conditions are described. The results of applying this theory to strength prediction are presented. Section 3 describes of modification of the classical theory of critical distance in cases accounting of elasto-plastic material behavior. The parameters of the Simplified Johnson Cook model are identified. The results of using elasto-plastic analysis for strength evaluation are presented.

2. Theory of Critical Distances based on linear-elastic analysis

2.1. The fundamentals of the classical theory of critical distances

According to the theory of critical distances the failure of notched components can be predicted using linear-elastic stress information in a critical region close to the notch tip. Failure initiates when applied stress provides an existence of area with stress higher ultimate stress of the material. In this work the spatial size of this area is determined according to Point method (Figure 1) of the Theory of Critical Distances [4]. The empirical relationship in terms of the Point Method to predict the fracture of laboratory samples with a complex geometry is represented by equation (1). The Point Method postulates that a notched component is failure when the effective stress σ_{eff} at a given distance from the stress raiser apex equals the material plain inherent strength.

$$\sigma_{eff} = \sigma_1 \left(r = L/2, \theta = 0 \right) = \sigma_0 \tag{1}$$

where r, θ - polar coordinates, σ_1 - is the range of the maximum principal stress, L - material characteristic length, which is determined according to the relationship:

$$L = \frac{1}{\pi} \left(\frac{K_{Ic}}{\sigma_0} \right)^2 \tag{2}$$

where K_{Ic} is the plane strain fracture toughness and σ_0 is the material inherent strength.

An experimental way to determine L is by testing samples containing notches of different sharpness is proposed [6]. This procedure is summarized in Figure 2. As postulated by the PM, the coordinates of the point at which the two linear-elastic stress-distance curves, plotted in the incipient failure condition, intersect each other allow us estimated directly the material length parameter L and inherent strength σ_0.
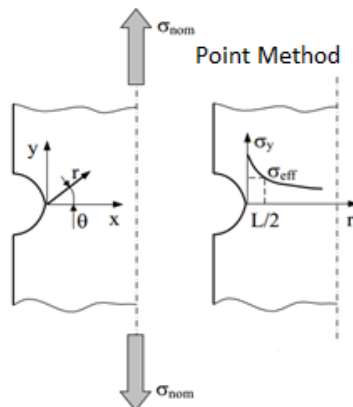
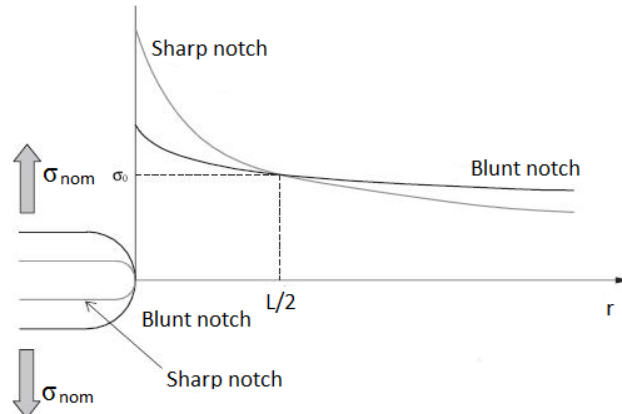**Figure 1.** Local systems of coordinates and effective stress according to the PM

Figure 2. Determination of length scale parameter L through experimental results.

2.2. *Materials and experimental conditions*

Thirty-three cylindrical samples of titanium alloy Grade2 were tested under tensile loading with different grippers travel speed, which ensured several orders of strain rate. Tensile tests were carried out using a 300 kN electromechanical testing machine Shimadzu AG-X Plus. Measurement of strain during materials testing was carried out using video extensometer TRViewX240S f12.5. For this study three types of cylindrical specimens were used with different stress concentrators such as semi-circular edge notches with radius 1 mm and 2 mm, V-shaped notches and un-notched (plain) specimens. For all specimens, the notches depth was kept constant. The geometry of specimens is shown in Figure 3. Loading tensile specimens occurred with range of strain rate 10^{-4} - 10 s^{-1}. During each test the failure force and time of failure were determined. The failure force was taken equal to the maximum force recorded during each test. The stress fields requiring to calculation of the effective stress according to definitions TCD were determined by solving Finite-Element models done by the finite-element package Abaqus SE.
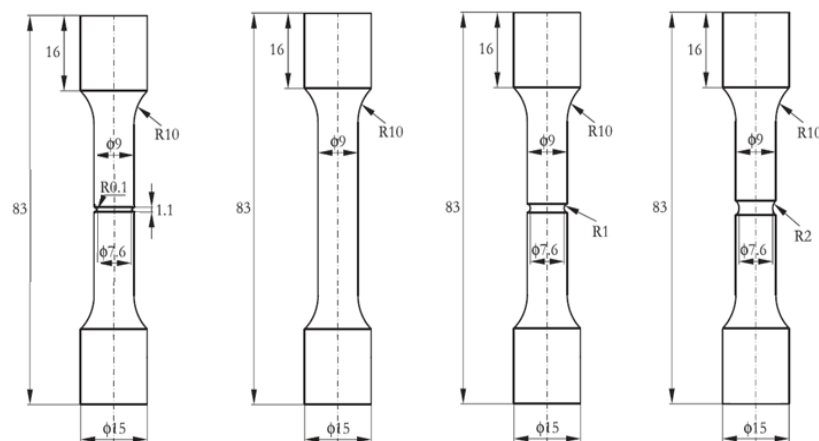

Figure 3. Geometry of specimens.

The results generated by testing un-notched and notched specimens titanium alloy Grade2 are summarized in Tables 1 in terms of failure force, F_f, time to failure, T_f, nominal loading rate, $\dot{F}$, and nominal strain rate $\dot{\varepsilon}_{nom}$.

Table 1. Summary of the experimental results generated by testing cylindrical samples of Grade2.

	r_n (mm)	Grip speed (mm/s)	F (kN)	T_f (s)	$\dot{F}$ (kN/s)	Displacement (mm)	$\dot{\varepsilon}_{nom}$ (1/s)
S 1	Plain	0.01	28.4	813.8	0.035	3.3939	0.0002
S 2	Plain	0.01	28.7	812.5	0.035	3.2740	0.0002
S 5	Plain	0.1	29.9	72.1	0.415	2.8747	0.002
S 6	Plain	0.1	29.8	70.9	0.420	2.7509	0.002
S 7	Plain	5	32.3	1.6	20.19	0.8956	0.051
S 8	Plain	5	32.5	1.6	20.31	0.7801	0.049
S 10	2	0.01	25.7	553	0.046	1.9571	0.003
S 11	2	0.01	26.2	550.3	0.048	1.9766	0.003
S 12	2	0.01	26	529.2	0.049	1.7608	0.003
S 13	2	0.1	27.4	50.3	0.545	1.5788	0.027
S 14	2	0.1	27.6	51.7	0.534	1.705	0.027
S 15	2	0.1	27.3	49.3	0.554	1.5962	0.027
S 16	2	5	29.9	1.1	27.182	0.2663	0.38
S 17	2	5	29.8	1.2	24.833	0.2991	0.427
S 18	2	5	29.9	1.1	27.182	0.2938	0.399
S 19	1	0.01	27.2	618.6	0.044	2.6024	0.0046
S 20	1	0.01	27	610	0.044	2.5759	0.0047
S 23	1	0.1	28.3	60.4	0.469	2.3252	0.0448
S 24	1	0.1	27.9	60	0.465	2.3752	0.0443
S 25	1	5	30.6	1.3	23.538	0.5421	0.884
S 26	1	5	30.7	1.4	21.929	0.6001	0.925
S 27	1	5	30.4	1.4	21.714	0.6041	1.08
S 28	0.1	0.01	27.13	641.6	0.043	2.7095	0.0078
S 29	0.1	0.01	27.2	585.8	0.046	2.2454	0.0078
S 30	0.1	0.1	28.3	59.9	0.472	2.4256	0.076
S 31	0.1	0.1	28.4	60.3	0.471	2.3841	0.077
S 32	0.1	5	31.1	1.3	23.923	0.4565	1.43
S 33	0.1	5	30.8	1.4	22	0.6893	1.8

2.3. Results

To use the Theory of Critical Distances to re-analyse the results generated by testing the notched cylindrical samples of titanium alloy Grade2, the initial assumption was made that the inherent strength could be taken equal to the corresponding plain material strength, that is:

$$\sigma_0\left(\dot{\varepsilon}_{nom}\right) = \sigma_f\left(\dot{\varepsilon}_{nom}\right) \tag{3}$$

According to the plain results reported in Table 1, $\sigma_f\left(\dot{\varepsilon}_{nom}\right)$ was expressed as follows:

$$\sigma_f\left(\dot{\varepsilon}_{nom}\right) = 539.22 \cdot \dot{\varepsilon}_{nom}^{0.0121} \text{ (MPa)} \tag{4}$$

The chart of Figure 4a shows the linear-elastic stress–distance curves plotted, under quasi-static loading ($\dot{\varepsilon}_{nom} = 0.0078\,\text{s}^{-1}$), in the incipient failure condition. As shown in Figure 4a, the use of the material ultimate tensile strength ($\sigma_{UTS} = 506.46$ MPa) resulted in a value for the critical distance equal to 1.702 mm. The same strategy (Figure 4b) was followed also to estimate the critical distance value under $\dot{\varepsilon}_{nom} = 1.43\,\text{s}^{-1}$ and resulted in a critical distance value of 2.734 mm. Therefore, the two critical distance values allowed us to obtain the following relationship:

$$L\left(\dot{\varepsilon}_{nom}\right) = 2.646 \cdot \dot{\varepsilon}_{nom}^{0.09095} \text{ (mm)} \tag{5}$$

By making use of power laws (5) the effective stress was then calculated, in the incipient failure condition, according to Point Method Theory of Critical Distances (equation 1). The results of this final re-analysis are summarized in Figure 4c, where the error is calculated as:

$$Error = \frac{\sigma_{eff}(\dot{\varepsilon}_{nom}) - \sigma_0(\dot{\varepsilon}_{nom})}{\sigma_0(\dot{\varepsilon}_{nom})} \qquad (6)$$
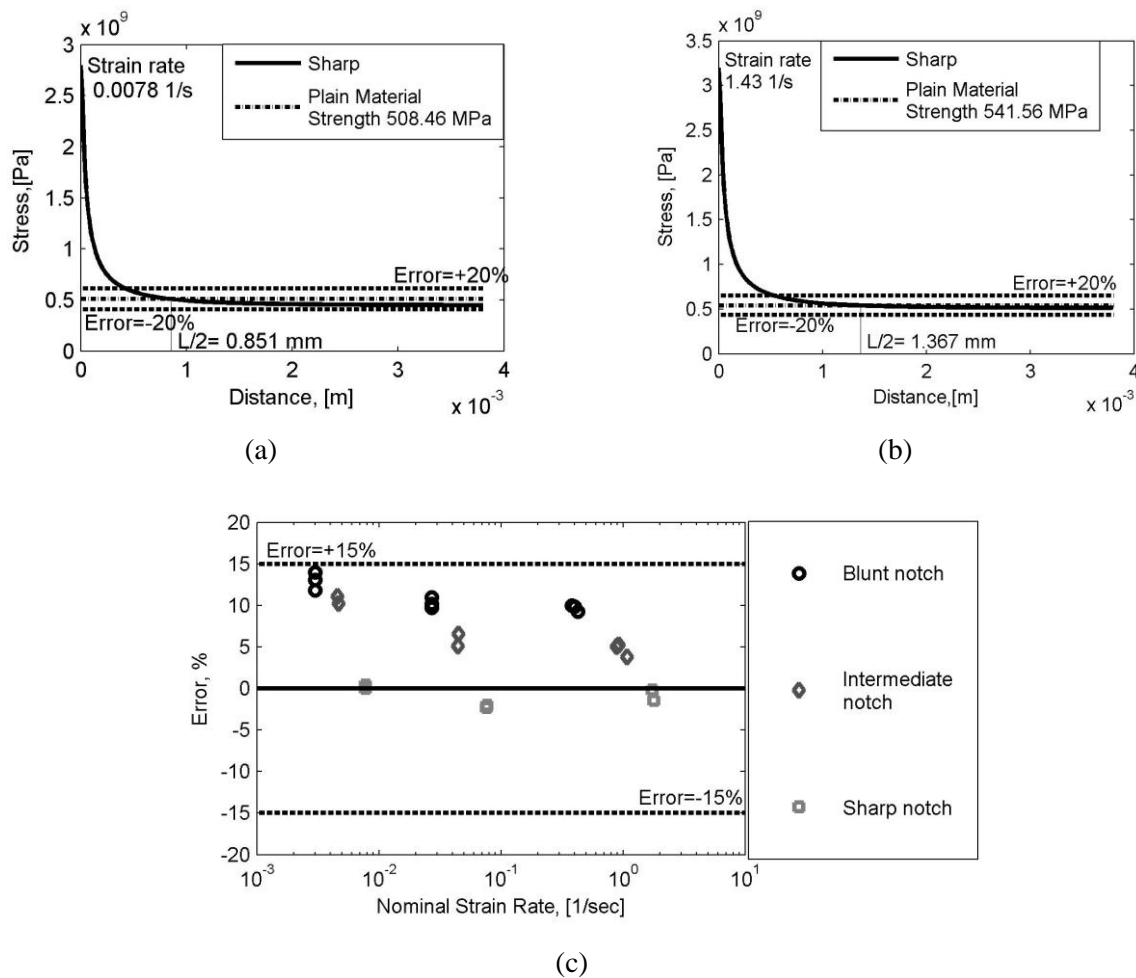

(a)

(b)

(c)

Figure 4. Local linear-elastic stress fields under $\dot{\varepsilon}_{nom} = 0.0078\,\mathrm{s}^{-1}$ and $\dot{\varepsilon}_{nom} = 1.43\,\mathrm{s}^{-1}$ (b); accuracy of the Point Method of TCD in predicting the strength (c and d).

The error diagrams reported in Figure 4c prove that the TCD was capable of accurately estimating also the strength of the notched specimens of titanium alloy Grade2, with the estimates falling within an error interval of ±15%.

3. Theory of Critical Distances based on elasto-plastic analysis

In this study considered the modification of the classical theory of critical distance in cases of elasto-plastic material behavior for improve the accuracy of estimation of lifetime of notched components, because the material behavior being, by nature, highly non-linear and cannot be described in the framework of the linear theory of elasticity. Accounting plasticity has been implemented with the use of the Simplified Johnson-Cook model.

3.1. The Simplified Johnson-Cook model

Johnson-Cook equation (7) consists of three brackets which defines the effect of the strain, strain rate and temperature on the flow curve of the tested material. The adiabatic heating effect is considered negligible for the tension tests as the material necks down at relatively

low strains before any significant adiabatic heating. The first bracket includes the parameters related with the flow curve at the reference strain rate. The first bracket $(A+B\varepsilon^n)$ is determined from the quasi-static tensile test at the reference strain rate of 0.0002 s^{-1}. The plastic strain is calculated by subtracting the elastic strain from the total strain as depicted in Figure 5.

$$\sigma = \left(A + B\varepsilon^n\right)\left(1 + C\ln\frac{\dot{\varepsilon}}{\dot{\varepsilon}_0}\right)\left(1 - \frac{T_m - T}{T_m - T_R}\right) \tag{7}$$

For the reference strain rate stress values are fitted with $(A+B\varepsilon^n)$ as shown in Figure 6. The fitting is performed until about 0.08 strain since above this strain level necking starts. The fitting parameters for the tensile, A, B and n are shown in the same graph: A=363.1 MPa, B=389.89 MPa and n=0.435. For the second parentheses which show the strain rate effect on the deformation, the yield stress values are drawn as function strain rate as shown in Figure 7. At last to check the reliability of the model parameters we compared with the real tests result with the model results as in Figure 8 model and experimental stress values are very much similar.
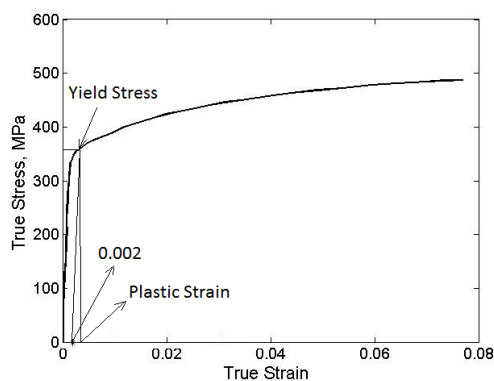

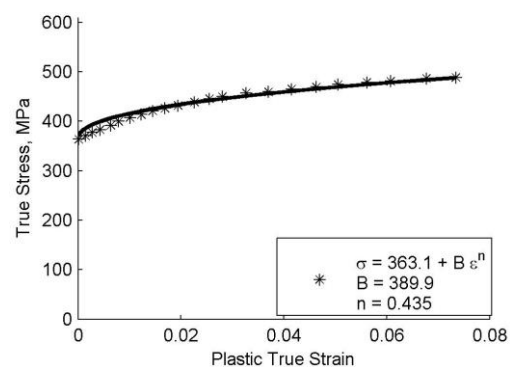**Figure 5.** True stress-strain curve of tensile test at reference strain rate.

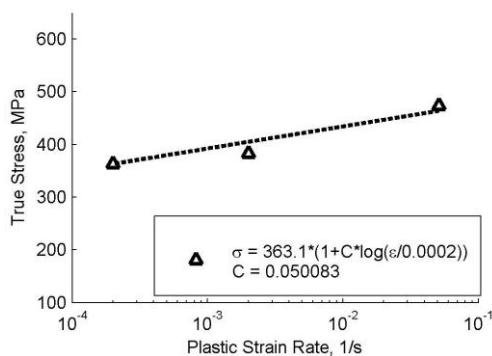**Figure 6.** True plastic stress versus true plastic strain.

Figure 7. Yield strength vs. logarithmic strain rate curve.
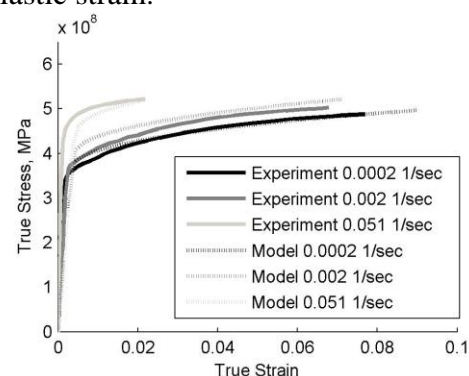

Figure 8. Johnson−Cook curve fitting result vs. experimental result.

3.2. The Theory of Critical Distances based on elasto-plastic analysis

The same analysis was carried out as in the previous section. However, for determine the value of the critical distance elasto-plastic stress fields were used. The chart of Figure 9 shows the elasto-plastic stress–distance curves plotted, under different strain rates. Determining the

value of the critical distance for different strain rates, we can see that, in contrast to linear-elastic analysis, the value of the critical distance is a constant equal 0.5 mm.
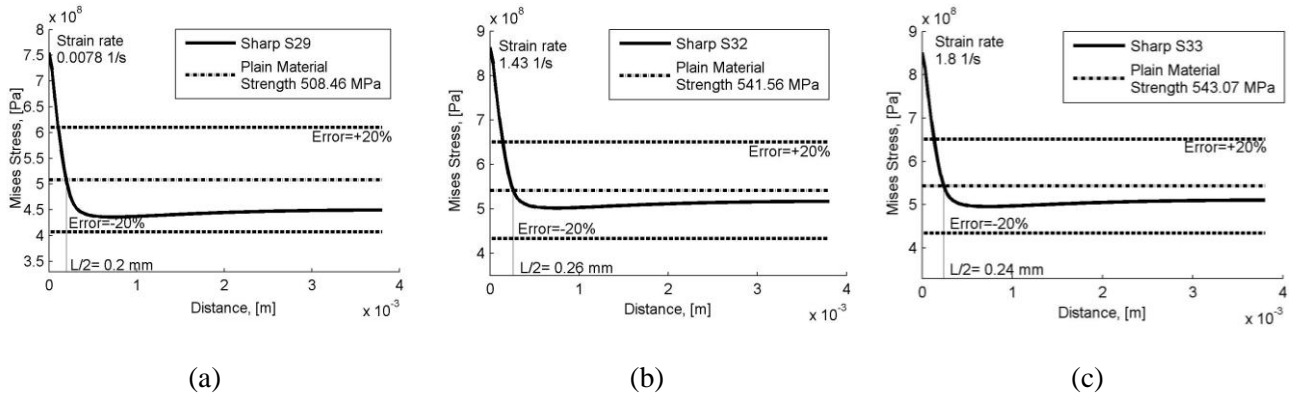

(a)   (b)   (c)

Figure 9. Local elasto-plastic stress fields under $\dot{\varepsilon}_{nom} = 0.0078\,\mathrm{s}^{-1}$ (a); $\dot{\varepsilon}_{nom} = 1.43\,\mathrm{s}^{-1}$ (b); $\dot{\varepsilon}_{nom} = 1.8\,\mathrm{s}^{-1}$ (c).

By making use value of critical distances the effective stress was then calculated, in the incipient failure condition, according to Point Method Theory of Critical Distances (equation 1). The results of this final re-analysis are summarized in Figure 10, where the error is calculated according equation (6):
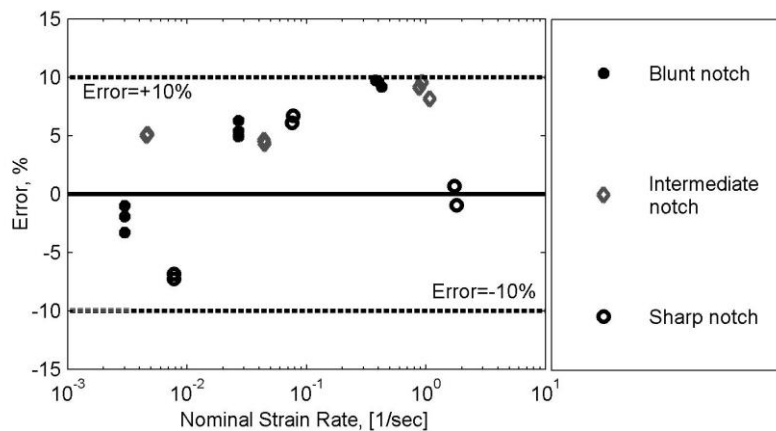

Figure 10. Accuracy of the TCD based on elasto-plastic analysis in predicting the strength.

4. Conclusions

In this study modification of the classical theory of critical distance in cases of elasto-plastic material behavior was proposed. Accounting plasticity has been implemented with the use of the Simplified Johnson-Cook model. The parameters for this model were identified from experimentally determined stress-strain curves by curve fitting techniques. The results of elasto-plastic analyses of stress distributions near a wide variety of notches are presented. The results showed that the use of the modification of the TCD based on elasto-plastic analysis gives us estimates falling within an error band of ±5-10%, that more accurate predictions than the linear elastic TCD solution. The use of an improved description of the stress-strain state at the notch tip allows introducing the critical distances as a material parameter.

Acknowledgments

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