IOP Conf. Series: Materials Science and Engineering **208** (2017) 012010 doi:10.1088/1757-899X/208/1/012010

Exact solutions for layered large-scale convection induced by tangential stresses specified on the free boundary of a fluid layer

N.V. Burmasheva[1,2,a], **E.Yu. Prosviryakov**[2,3,b]

[1] Ural Federal University, 19 Mira Street, Ekaterinburg, 620002, Russia
[2] Institute of Engineering Science, 34 Komsomolskya Street, Ekaterinburg, 620049, Russia
[3] Kazan National Research Technical University, 10 K.Marx Street, Kazan, 420111, Russia

E-mail: [a]nat_burm@mail.ru, [b]evgen_pros@mail.ru

Abstract. A new exact solution for layered convection of a viscous incompressible fluid is found in this paper. A fluid flow in an infinite layer is considered. Convection in the fluid is induced by tangential stresses specified on the upper non-deformable boundary. Temperature corrections are given on the both boundaries of the fluid layer. The analysis of hydrodynamic fields allows us to state the presence of two stagnant points in the flow of a fluid. It is shown that, in the case of thermocapillary convection in a fluid, only one stagnation point can exist.

1. Introduction

A number of exact solutions describing nonisothermal fluid flows have been obtained to date. A great majority of exact nonisothermal solutions of the Navier-Stokes equations in the Boussinesq approximation are formulated in terms of velocity, pressure, and temperature fields that depend linearly on horizontal coordinate [1, 2]. In refs. [1, 2, 3, 4, 5, 6, 7] one can find the most important papers and reviews that follow fundamental studies authored by Ostroumov [8] and Birikh [9] and deal with physically meaningful exact solutions for advective and convective flows. In this paper, we will consider new exact solutions for layered convection under induced by tangential forces on the upper boundary of an infinite fluid layer.

2. Boundary value problem formulation

We consider convective large-scale motions of a viscous incompressible fluid in an infinite horizontal strip. Mathematical simulation of large-scale motions in a fluid is based on the use of layered flows ($V_z \equiv 0$) [1, 2, 3, 4, 5, 6]. Convective flows of a viscous incompressible fluid are described by the Oberbeck-Boussinesq equation system [7, 10], which consists of the Navier-Stokes equations

$$(\mathbf{V} \cdot \nabla)\mathbf{V} = -\nabla P + \nu \triangle \mathbf{V} + g\beta T\mathbf{k}, \tag{1}$$

the heat equation

$$\mathbf{V} \cdot \nabla T = \chi \triangle T, \tag{2}$$

and the incompressibility equation

$$\nabla \cdot \mathbf{V} = 0. \tag{3}$$

Here $\mathbf{V}(x, y, z) = (V_x, V_y, V_z)$ is a velocity vector; P is the deviation of pressure from hydrostatic, divided by constant average fluid density ρ; T is the deviation from the average temperature; ν and χ are the coefficients of the kinematic viscosity and thermal diffusivity of the fluid, respectively; $\mathbf{k}$ is the unit vector of the z axis directed vertically upwards; ∇ is the Hamiltonian operator; $\triangle$ is the three-dimensional Laplace operator.

Note that, although the terms of the convective derivative from the Navier-Stokes equation and the incompressibility equation are identically equal to zero, they remain present in the heat equation. Therefore, strictly speaking, a nonlinear problem is solved.

To find the hydrodynamic fields, we represent the velocities in the following form [1, 2]:

$$V_x = u(z), \quad V_y = v(z). \tag{4}$$

This representation allows us to find a solution for an overdetermined system. The system (1) - (3) consists of 5 equations with respect to four unknowns (velocity components V_x, V_y and physical fields T and P). The solution (4) identically satisfies the incompressibility equation (3). Thus, the velocity field (4) is solenoidal.

The pressure and the temperature can be represented in special linear forms depending on the horizontal coordinates as

$$T = T_0(z) + T_1(z)x + T_2(z)y, \quad P = P_0(z) + P_1(z)x + P_2(z)y. \tag{5}$$

Then we substitute the selected class of solutions (4) - (5) into the Navier-Stokes equation (1) and the heat equation (2). Equating the coefficients at the identical powers of the horizontal coordinates x and y, we obtain the following system of ordinary differential equations:

$$T_1'' = 0, \qquad T_2'' = 0. \tag{6}$$

$$\frac{\partial P_1}{\partial z} = g\beta T_1, \qquad \frac{\partial P_2}{\partial z} = g\beta T_2 \tag{7}$$

$$\nu u'' = P_1, \qquad \nu v'' = P_2, \tag{8}$$

$$\chi T_0'' = uT_1 + vT_2, \tag{9}$$

$$\frac{\partial P_0}{\partial z} = g\beta T_0. \tag{10}$$

In this paper, we assume that the lower boundary of the fluid layer is absolutely rigid and fixed. The upper boundary is assumed to be free and undeformed. We consider the following boundary conditions: at the lower boundary ($z = 0$) of the fluid layer the adhesion condition is satisfied and the temperature is given by the function

$$T = Ax + By. \tag{11}$$

On the upper boundary ($z = h$), constant pressure S acts and the temperature is given by the function

$$T = \vartheta + Cx + Dy. \tag{12}$$

In addition, stresses are given on the free boundary $z = h$ as

$$\eta\frac{du}{dz} = \xi_1, \quad \eta\frac{dv}{dz} = \xi_2. \tag{13}$$

Thus, we obtain the following system of boundary conditions:

$$u(0) = v(0) = 0,$$

$$\begin{array}{lll} T_0(0) = 0, & T_1(0) = A, & T_2(0) = B, \\ T_0(h) = \vartheta, & T_1(h) = C, & T_2(h) = D, \\ P_0(h) = S, & P_1(h) = 0, & P_2(h) = 0, \\ & \eta \dfrac{du}{dz}(h) = \xi_1, & \\ & \eta \dfrac{dv}{dz}(h) = \xi_2. & \end{array} \tag{14}$$

Without loss of generality, we can assume that $S = 0$, thereby counting the reduced pressure from the level specified at the upper boundary.

3. Equation system solution

The case of temperature and stresses simultaneously set on the upper boundary was considered earlier in [11]. Therefore, another particular case of the above-mentioned boundary value problem is considered here, namely, when the temperature perturbation is specified at the lower boundary ($C = D = 0$).

Integrating the system of equations (6) - (10) in view of the boundary conditions, we obtain the exact solution

$$u(z) = \frac{z(24h\xi_1\nu + 6B\beta\eta g h^2 z - 8B\beta\eta g h z^2 + 3B\beta\eta g z^3)}{24\eta h\nu}, \tag{15}$$

$$v(z) = \frac{z(24h\xi_2\nu + 6A\beta\eta g h^2 z - 8A\beta\eta g h z^2 + 3A\beta\eta g z^3)}{24\eta h\nu}, \tag{16}$$

$$\begin{aligned} P_0(z) = {} & \frac{168h\nu(60\chi\eta(2hS - \beta g h^2\vartheta + \beta g\vartheta z^2) + B\beta g\xi_2(2h^5 - 5h^3z^2 + 5hz^4 - 2z^5))}{20160\chi\eta h^2\nu} + \\ & + \frac{A\beta g(168h\xi_1\nu(2h^5 - 5h^3z^2 + 5hz^4 - 2z^5))}{20160\chi\eta h^2\nu} + \\ & + \frac{B\beta\eta g(35h^8 - 80h^6z^2 + 168h^3z^5 - 196h^2z^6 + 88hz^7 - 15z^8)}{20160\chi\eta h^2\nu}, \end{aligned} \tag{17}$$

$$P_1(z) = -\frac{A\beta g(h-z)^2}{2h}, \tag{18}$$

$$P_2(z) = -\frac{B\beta g(h-z)^2}{2h}, \tag{19}$$

$$\begin{aligned} T_0(z) = -z \Bigg(& \frac{210h\nu(-12\chi\eta\vartheta + B\xi_2(h^3 - 2hz^2 + z^3))}{2520\chi\eta h^2\nu} + \\ & + \frac{A(210h\xi_1\nu(h^3 - 2hz^2 + z^3) + B\beta\eta g(20h^6 - 105h^3z^3 + 147h^2z^4 - 77hz^5 + 15z^6))}{2520\chi\eta h^2\nu} \Bigg), \end{aligned} \tag{20}$$

$$T_1(z) = A - \frac{Az}{h}, \tag{21}$$

$$T_2(z) = B - \frac{Bz}{h}. \tag{22}$$

Then we formulate the problem of finding the number of stratification points in the velocity, pressure and temperature fields. Note that the case $A = B = 0$ is not considered here due to its triviality. Indeed, if $A = B = 0$, then $P_1 = P_2 = T_1 = T_2 \equiv 0$, $u = \dfrac{z\xi_1}{\eta}$, $v = \dfrac{z\xi_2}{\eta}$, $P_0(z) = \dfrac{\beta g\vartheta\chi\eta(-h^2+z^2)}{2h}$, $T_0(z) = \dfrac{z\vartheta}{h}$, and this means that there are no stratifications. Therefore, we assume hereinafter that $A^2 + B^2 \neq 0$.

4. Investigation of velocity components

The velocity components $uandv$ are zero at the origin (the factor z is explicit). Let us now ask the question how many more zeros these functions can have. This question is quite important, since the presence of zeros in these functions indicates the existence of counterflows (stratifications) in the convective flow of a viscous incompressible fluid.

We normalize the expressions in (3) for the components $u(z)$ and $v(z)$ of the velocity vector $\mathbf{V}$ by A reducing them to a dimensionless form. Without loss of generality, we set $A \neq 0$.

We introduce the following dimensionless parameters: $\gamma = B/A$, $\delta = h/l$, where h is the characteristic vertical dimension of the layer and l is the characteristic horizontal dimension of the layer. In addition, the dimensionless coordinate z is determined as $z \to z/h$.

The velocities $u(z)$ and $v(z)$ are divided by $\dfrac{g\beta A l^3}{\nu}$. As a result, we finally arrive at the formula

$$u = \frac{z\delta^3}{24}\left(\gamma(6z - 8z^2 + 3z^3) + 24W_1\right), \tag{23}$$

$$v = \frac{z\delta^3}{24}\left(6z - 8z^2 + 3z^3 + 24W_2\right). \tag{24}$$

Here, $W_i = \dfrac{\nu\xi_i}{g\beta C\eta h^2}$ is the Weber number determined for the value ξ_i.

We study the function $u(z)$. It obviously vanishes at the point $z = 0$.

If $\gamma = 0$, then this zero is unique for $W_1 \neq 0$. If $\gamma^2 + W_1^2 = 0$, then the function $u(z) \equiv 0$. Therefore, we set $\gamma \neq 0$ and represent the function $u(z)$ in the form

$$u(z) = \frac{z\delta^3\gamma}{24} f(z, 24W_1/\gamma), \quad f(z,q) = 6z - 8z^2 + 3z^3 + q. \tag{25}$$

The number of the zeros of the function u_1 on the interval $(0;1)$ determines the number of stratifications of the velocity field $u(z)$.

Figure 1 presents the plot of the function $f(z,0)$. This function reaches its maximum at the point $z_0 = (8-\sqrt{10})/9$ and takes the value $f_{max} = 4(68+5\sqrt{10})/243$ at this point. The function $f(z,0)$ has no zeros in the interval $(0;1)$; however, one can change the value of the parameter q so that the function $f(z,q)$ (as well as the function $u(z)$) will have one or even two zeros on the interval $(0;1)$.

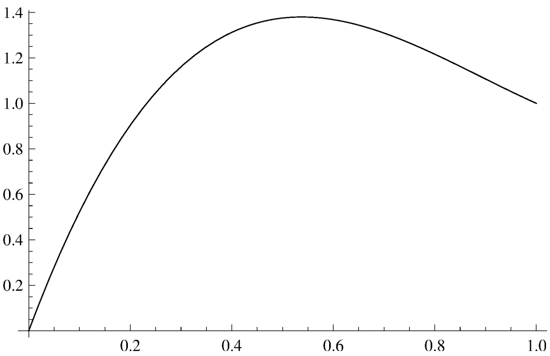

Figure 1. The plot of the function $f(z, 0)$.

Thus, returning to the function $u(z)$, we obtain the following estimates:

1) $\dfrac{24W_1}{\gamma} > f(0,0) = 0$, i. e. the function $u(z)$ does not change its sign;
2) $0 = f(0,0) > \dfrac{24W_1}{\gamma} \geq -f(1,0) = 1$, i. e. the function $u(z)$ changes its sign once;
3) $1 > \dfrac{24W_1}{\gamma} > -f(1,0) = -f_{max}$, i. e. the function $u(z)$ changes its sign twice;
4) $-f_{max} = -f(1,0) = \dfrac{24W_1}{\gamma}$, i. e. the function $u(z)$ changes its sign once;
5) $\dfrac{24W_1}{\gamma} < -f_{max}$, i. e. the function $u(z)$ does not change its sign.

The behavior of the function $u(z)$ for various values of the ratio W_1/γ is shown in figure 2.

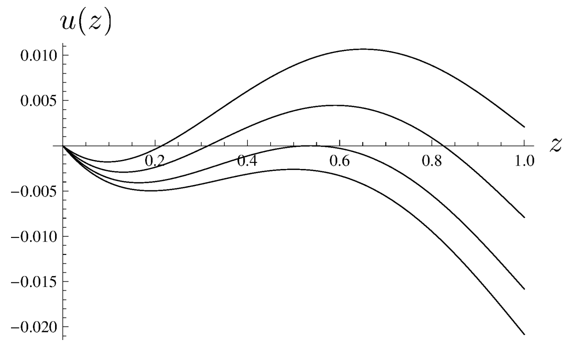

Figure 2. The behavior of the function $u(z)$.

Similar estimates are valid for the function $v(z)$:

1) $24W_2 > f(0,0) = 0$, i. e. the function $v(z)$ does not change the sign;
2) $0 = f(0,0) > 24W_2 \geq -f(1,0) = 1$, i. e. the function $v(z)$ changes its sign once;
3) $1 > 24W_2 > -f(1,0) = -f_{max}$, i. e. the function $v(z)$ changes its sign twice;
4) $-f_{max} = -f(1,0) = 24W_2$, i. e. the function $v(z)$ changes its sign once;
5) $24W_2 < -f_{max}$, i. e. the function $v(z)$ does not change the sign.

Of particular interest in this situation is the study of the velocity vector hodograph. Theoretically, there cannot be more than 25 different forms of the hodograph (5 variants for the $u(z)$ curve and 5 variants for the $v(z)$ curve). However, there are situations that generate one and the same hodograph (for example, for $\gamma = 1$ when $W_1 = W_2 = -1$ or $W_1 = W_2 = 0$, the hodograph curve is the bissectrice of the first quadrant). In addition, we should not forget about the relationship between the parameters of the problem ($W_1\xi_2 = W_2\xi_1$), which leads to additional constraints and, consequently, to a decrease in the number of fundamentally different trajectories of the velocity vector hodograph. As an example, in figures 3 and 4 the velocity vector hodographs are given for two fixed values of the ratio W_1/γ.

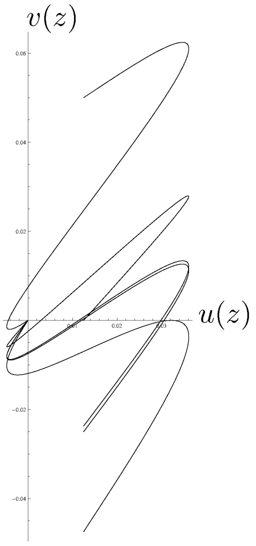

Figure 3. The velocity vector hodograph.

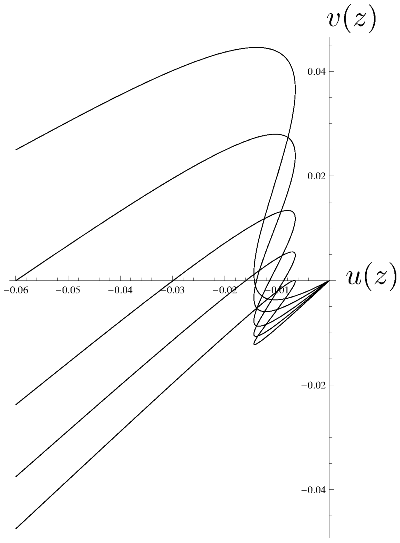

Figure 4. The velocity vector hodograph.

Let us write out the basic quantitative measure of vorticity:

$$\vec{\Omega} = rot\mathbf{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & 0 \end{vmatrix}, \tag{26}$$

or coordinatewise:

$$\Omega_x = -\frac{\partial v}{\partial z} = -\frac{\delta^3}{2}(2W_2 + z - 2z^2 + z^3), \quad \Omega_y = \frac{\partial u}{\partial z} = \frac{\delta^3}{2}(2W_1 + \triangle(z - 2z^2 + z^3)), \quad \Omega_z = 0. \tag{27}$$

Of particular interest here are combinations of parameters for which the vorticity becomes zero at least at several points (it is not identically equal to zero). It is at these points that the direction of the vortex changes.

It is obvious that, for $\gamma = 0$, there are no such points for the component Ω_y; therefore, we assume that $\gamma \neq 0$. Then Ω_2 can be rewritten as

$$\Omega_y = -\frac{\delta^3}{2}(2W_1 + \triangle(z - 2z^2 + z^3)) = -\frac{\delta^3\gamma}{2}(2W_1/\gamma + \triangle(z - 2z^2 + z^3)). \tag{28}$$

That is, the number of points at which the components $\Omega_x and \Omega_y$ change their direction (one at a time or simultaneously) is governed by the number of zeros of the function$g(z, c) = c + z - 2z^2 + z^3$ (when $c = 2W_1/\triangle\gamma$ for the component Ω_y and when $c = 2W_2$ for Ω_x). With all the further calculations, one can refer to the behavior of the function $g(z, c)$, whose plot for $c = 0$ is presented in figure 5.

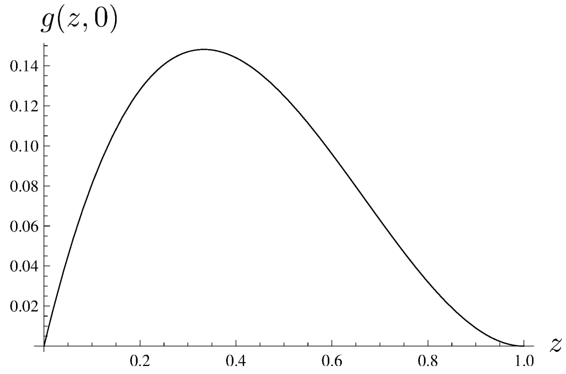

Figure 5. The plot of the function $g(z, 0)$.

5. Conclusion

For the discussed boundary-value problem of simultaneous setting of the boundary temperature and stresses, it has been shown that, in the hydrodynamic fields (in the velocity field in particular), stratifications may arise, depending on certain combinations of the parameters specifying the physical quantities at the boundaries of the region under study. The number of stratification points is always different (0, 1 or 2), and it depends on the values of the constants. The same values also determine the positions of stratification points in the region under investigation. In addition, it has been demonstrated that the motion is vortex (the vorticity is not identically zero); however, under certain combinations of the system parameters, the vortex may change its direction. The conditions for this phenomenon have been discussed.

Acknowledgments

This work was supported by the Foundation for Assistance to Small Innovative Enterprises in Science and Technology (the UMNIK program).

References

[1] Aristov S N and Prosviryakov E Yu 2013 *Rus. J. Nonlin. Dyn.* **9** 651
[2] Aristov S N and Prosviryakov E Yu 2016 *Theor. Found. Chem. Eng.* **50** 286
[3] Sidorov A F 1989 *J. Appl. Mech. Tech. Phy.* **30** 197
[4] Shvarz K G 2014 *Fluid Dynamics* 4 438
[5] Aristov S N, Knyazev D V and Polyanin A D 2009 *Theor. Found. Chem. Eng.* **43** 642
[6] Andreev V K and Bekezhanova V B 2013 *J. Appl. Mech. Tech. Phy.* **54** 171
[7] Gershuni G Z and Zhukhovitskii E M 1976 *Convective stability of incompressible liquid* (Jerusalem: Wiley)
[8] Ostroumov G A 1958 *Free convection under the condition of the internal problem* (NASA TM)
[9] Birich R V 1966 *J. Appl. Mech. Tech. Phy.* **3** 69
[10] Landau L D and Lifshitz E M 1987 *Fluid Mechanics* (Oxford: PergamonPress)
[11] Burmasheva N V and Prosviryakov E Yu 2017 *Vestn. Samar. Gos. Tekh. Univ., Ser. Fiz. Mat. Nauki (J. Samara State Tech. Univ., Ser. Phys. and Math. Sci.)* **21** (*Preprint* doi: 10.14498/vsgtu1527)