

Kinematic dynamo in a tetrahedron composed of helical Fourier modes

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Abstract. It is generally believed that helicity can play a significant role in turbulent systems, e.g. supporting the generation of large-scale magnetic fields, but its exact contribution is not clearly understood. For example there are well-known examples of large scale dynamos produced by a flow which is pointwise non-helical. In any case a break of mirror symmetry seems to be always at the heart of the dynamo mechanism. A fruitful framework to analyze such processes is the use of helical mode decomposition. In pure hydrodynamics such framework has proved its availability in study of the processes responsible for helicity cascades. It has also been used in the analysis of MHD helical mode interactions. The present work deals with the kinematic dynamo problem, solving the induction equation within the framework of helical Fourier modes decomposition. We show that the simplest modes configuration leading to an unstable solution has the form of a tetrahedron. Then the dynamo is produced by only two scales flow. We find necessary conditions for such dynamo action, not certainly related to flow helicity. The results help to understand generic dynamo flows like the one studied by G.O. Roberts (1972).

1. Introduction

It is generally believed that helicity can play a significant role in turbulent systems, e.g. supporting the generation of large-scale magnetic fields, but its exact contribution is not clearly understood. For example there are well-known examples of large scale dynamos produced by a flow which is pointwise non-helical but in any case a break of mirror symmetry seems to be always at the heart of the dynamo mechanism. Most of the dynamo studies are based on solutions of the mean field equations, direct numerical simulations or shell model of turbulence. Another framework to analyze such processes is the use of helical mode decomposition. In pure hydrodynamics such framework provide comprehensive studies the processes responsible for helicity cascades [1–4]. It has also been used in the analysis of MHD mode interactions [5].

The present work deals with the kinematic dynamo problem, solving the nondiffusive induction equation within the framework of helical Fourier modes decomposition. We look for the simplest modes configuration leading to an unstable solution with only two flow scales. We look for necessary conditions for the dynamo action in terms of helical mode amplitudes, not directly related to flow helicity. The results should help for better understanding generic dynamo flows like the one studied by G.O. Roberts [6].

2. Helical decomposition

The induction equation that governs the time evolution of the magnetic induction \mathbf{b} for a given flow \mathbf{u} is

$$(\partial_t - \eta \nabla^2) \mathbf{b}(\mathbf{x}) = \nabla \times (\mathbf{u}(\mathbf{x}) \times \mathbf{b}(\mathbf{x})), \quad (1)$$



where η is a magnetic diffusivity. Assuming triply periodic boundary conditions in a cube of volume L^3 , both fields, flow and magnetic induction, can be expanded into discrete Fourier series:

$$\mathbf{u}(\mathbf{x}) = \sum_{\mathbf{k}} \mathbf{u}(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}}, \quad \mathbf{b}(\mathbf{x}) = \sum_{\mathbf{k}} \mathbf{b}(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}}, \quad (2)$$

where $\mathbf{u}(\mathbf{k})$ and $\mathbf{b}(\mathbf{k})$ are the complex Fourier coefficients. Then in Fourier space the induction equation takes the form

$$(\partial_t + \eta k^2) \mathbf{b}(\mathbf{k}) = \sum_{\substack{\mathbf{p}, \mathbf{q} \\ \mathbf{k} + \mathbf{p} + \mathbf{q} = 0}} i \mathbf{k} \times (\mathbf{u}^*(\mathbf{p}) \times \mathbf{b}^*(\mathbf{q})). \quad (3)$$

Following the approach presented by [7] one can introduce, in Fourier space, a base of polarised helical waves \mathbf{h}^\pm defined as the eigenvectors of the curl operator [8–10],

$$i \mathbf{k} \times \mathbf{h}^\pm = \pm k \mathbf{h}^\pm. \quad (4)$$

Note that the helical vectors $\mathbf{h}^\pm(\mathbf{k})$ are defined up to an arbitrary rotation of axis \mathbf{k} . Taking

$$\mathbf{h}^\pm(\mathbf{k}) = \mathbf{u}_2(\mathbf{k}) \pm i \mathbf{u}_1(\mathbf{k}) \quad (5)$$

with $\mathbf{u}_1(\mathbf{k}) = (\mathbf{z}_k \times \mathbf{k}) / |\mathbf{z}_k \times \mathbf{k}|$ and $\mathbf{u}_2(\mathbf{k}) = \mathbf{u}_1(\mathbf{k}) \times \mathbf{k} / k$, where \mathbf{z}_k is an arbitrary vector that, in general, may depend on \mathbf{k} , though it is not proportional to \mathbf{k} . This approach has been extended to MHD with the following line of argument [11].

The Fourier modes of velocity and magnetic fields are expanded on that helical base

$$\mathbf{u}(\mathbf{k}) = u^+(\mathbf{k}) \mathbf{h}^+(\mathbf{k}) + u^-(\mathbf{k}) \mathbf{h}^-(\mathbf{k}), \quad (6)$$

$$\mathbf{b}(\mathbf{k}) = b^+(\mathbf{k}) \mathbf{h}^+(\mathbf{k}) + b^-(\mathbf{k}) \mathbf{h}^-(\mathbf{k}). \quad (7)$$

Replacing the expressions (6-7) for \mathbf{u} and \mathbf{b} in Eqs. (3), and projecting on to the helical base $\mathbf{h}^{s_k}(\mathbf{k})$ (with $s_k = \pm 1$) leads to the following equation

$$(d_t + \eta k^2) b^{s_k}(\mathbf{k}) = -s_k k \sum_{\Delta_{\mathbf{k}\mathbf{p}\mathbf{q}}} \sum_{s_p, s_q = \pm 1} g_{s_k, s_p, s_q}(\mathbf{k}, \mathbf{p}, \mathbf{q}) (u^{s_p}(\mathbf{p}) b^{s_q}(\mathbf{q}) - b^{s_p}(\mathbf{p}) u^{s_q}(\mathbf{q}))^*, \quad (8)$$

where $\Delta_{\mathbf{k}\mathbf{p}\mathbf{q}}$ means a sum over all possible triads ($\mathbf{k} + \mathbf{p} + \mathbf{q} = 0$) containing \mathbf{k} and two arbitrary vectors \mathbf{p} and \mathbf{q} , g is a function of $\mathbf{k}, \mathbf{p}, \mathbf{q}$, s_k, s_p, s_q is defined as

$$g_{s_k, s_p, s_q}(\mathbf{k}, \mathbf{p}, \mathbf{q}) \equiv -\frac{1}{\mathbf{h}^{s_k}(\mathbf{k})^* \cdot \mathbf{h}^{s_k}(\mathbf{k})} (\mathbf{h}^{s_k}(\mathbf{k})^* \times \mathbf{h}^{s_p}(\mathbf{p})^*) \cdot \mathbf{h}^{s_q}(\mathbf{q})^*. \quad (9)$$

Considering a single triadic interaction, it is not necessary to introduce an arbitrary unit vector \mathbf{z}_k to define the unit vectors \mathbf{u}_1 and \mathbf{u}_2 . Indeed, there is a natural direction which is represented by the unit vector perpendicular to the plane of the triad:

$$\boldsymbol{\lambda} = (\mathbf{k} \times \mathbf{p}) / |\mathbf{k} \times \mathbf{p}| = (\mathbf{p} \times \mathbf{q}) / |\mathbf{p} \times \mathbf{q}| = (\mathbf{q} \times \mathbf{k}) / |\mathbf{q} \times \mathbf{k}|. \quad (10)$$

A second unit vector $\boldsymbol{\mu}_k = \mathbf{k} \times \boldsymbol{\lambda} / k$ is introduced and the helical vectors are then defined as

$$\mathbf{h}^{s_k}(\mathbf{k}) = e^{i s_k \varphi_k} (\boldsymbol{\lambda} + i s_k \boldsymbol{\mu}_k). \quad (11)$$

The angle φ_k defines the rotation around \mathbf{k} needed to transform the base $(\boldsymbol{\mu}_k, \boldsymbol{\lambda})$ onto the base $(\mathbf{u}_1(\mathbf{k}), \mathbf{u}_2(\mathbf{k}))$. Since the base $(\boldsymbol{\mu}_k, \boldsymbol{\lambda})$ depends on the triad, the angle φ_k is also a function of $(\mathbf{k}, \mathbf{p}, \mathbf{q})$. The coupling constant for this triad then simply reduces to

$$g_{s_k, s_p, s_q} = -\frac{1}{2} e^{-i(s_k \varphi_k + s_p \varphi_p + s_q \varphi_q)} s_k s_p s_q (s_k \sin \alpha_k + s_p \sin \alpha_p + s_q \sin \alpha_q) \quad (12)$$

with

$$\sin \alpha_k = \frac{Q_{kpq}}{2 p q} \quad \sin \alpha_p = \frac{Q_{kpq}}{2 k q} \quad \sin \alpha_q = \frac{Q_{kpq}}{2 k p}, \quad (13)$$

and $Q_{kpq} = \sqrt{2 k^2 p^2 + 2 q^2 p^2 + 2 q^2 k^2 - k^4 - q^4 - p^4}$.

3. Solution in one triad

For one triad $\Delta_{\mathbf{k}\mathbf{p}\mathbf{q}}$ where \mathbf{k} , \mathbf{p} and \mathbf{q} are fixed, and using the notation u^{s_k} for $u^{s_k}(\mathbf{k})$, u_k^\pm for $u^\pm(\mathbf{k})$, and g_{s_k, s_p, s_q} for $g_{s_k, s_p, s_q}(\mathbf{k}, \mathbf{p}, \mathbf{q})$, the diffusionless induction equation is given by

$$\begin{aligned} d_t b^{s_k} &= -s_k k \sum_{t_p, t_q = \pm 1} g_{s_k, t_p, t_q} (u^{t_p} b^{t_q} - b^{t_p} u^{t_q})^* \\ d_t b^{s_p} &= -s_p p \sum_{t_q, t_k = \pm 1} g_{t_k, s_p, t_q} (u^{t_q} b^{t_k} - b^{t_q} u^{t_k})^* \\ d_t b^{s_q} &= -s_q q \sum_{t_k, t_p = \pm 1} g_{t_k, t_p, s_q} (u^{t_k} b^{t_p} - b^{t_k} u^{t_p})^*. \end{aligned} \quad (14)$$

which is equivalent to

$$\begin{aligned} d_t b_k^+ &= -k[g_{+,+,+}(u_p^+ b_q^+ - b_p^+ u_q^+)^* + g_{+,-,-}(u_p^- b_q^- - b_p^- u_q^-)^* \\ &\quad + g_{+,+,-}(u_p^+ b_q^- - b_p^+ u_q^-)^* + g_{+,-,+}(u_p^- b_q^+ - b_p^- u_q^+)^*] \\ d_t b_p^+ &= -p[g_{+,+,+}(u_q^+ b_k^+ - b_q^+ u_k^+)^* + g_{-,-,+}(u_q^- b_k^- - b_q^- u_k^-)^* \\ &\quad + g_{-,+,-}(u_q^+ b_k^- - b_q^+ u_k^-)^* + g_{+,-,+}(u_q^- b_k^+ - b_q^- u_k^+)^*] \\ d_t b_q^+ &= -q[g_{+,+,+}(u_k^+ b_p^+ - b_k^+ u_p^+)^* + g_{-,-,+}(u_k^- b_p^- - b_k^- u_p^-)^* \\ &\quad + g_{+,+,-}(u_k^+ b_p^- - b_k^+ u_p^-)^* + g_{+,-,+}(u_k^- b_p^+ - b_k^- u_p^+)^*] \\ d_t b_k^- &= +k[g_{-,-,+}(u_p^+ b_q^+ - b_p^+ u_q^+)^* + g_{-,-,-}(u_p^- b_q^- - b_p^- u_q^-)^* \\ &\quad + g_{-,+,-}(u_p^+ b_q^- - b_p^+ u_q^-)^* + g_{-,-,+}(u_p^- b_q^+ - b_p^- u_q^+)^*] \\ d_t b_p^- &= +p[g_{-,+,-}(u_q^+ b_k^+ - b_q^+ u_k^+)^* + g_{-,-,-}(u_q^- b_k^- - b_q^- u_k^-)^* \\ &\quad + g_{-,-,+}(u_q^+ b_k^- - b_q^+ u_k^-)^* + g_{+,-,-}(u_q^- b_k^+ - b_q^- u_k^+)^*] \\ d_t b_q^- &= +q[g_{-,+,-}(u_k^+ b_p^+ - b_k^+ u_p^+)^* + g_{-,-,-}(u_k^- b_p^- - b_k^- u_p^-)^* \\ &\quad + g_{+,-,-}(u_k^+ b_p^- - b_k^+ u_p^-)^* + g_{-,-,+}(u_k^- b_p^+ - b_k^- u_p^+)^*]. \end{aligned} \quad (15)$$

We look for a growing solution of the linear system of ODEs (15) in the form

$$b_k^{s_k} = \check{b}_k^{s_k} e^{\gamma t}, \quad \gamma = \pm \sqrt{a}. \quad (16)$$

Therefore a necessary condition for having a positive growthrate γ is to have a positive.

3.1. Single helical triad

In this case we assume $b^{-s_k} = b^{-s_p} = b^{-s_q} = u^{-s_k} = u^{-s_p} = u^{-s_q} = 0$. Then we solve an eigenvalue problem with three variables

$$\begin{aligned} d_t b^{s_k} &= -s_k k g_{s_k, s_p, s_q} (u^{s_p} b^{s_q} - b^{s_p} u^{s_q})^* \\ d_t b^{s_p} &= -s_p p g_{s_k, s_p, s_q} (u^{s_q} b^{s_k} - b^{s_q} u^{s_k})^* \\ d_t b^{s_q} &= -s_q q g_{s_k, s_p, s_q} (u^{s_k} b^{s_p} - b^{s_k} u^{s_p})^*. \end{aligned} \quad (17)$$

It leads to non trivial solutions

$$a = -s_k s_p s_q k p q |g_{s_k, s_p, s_q}|^2 F \quad (18)$$

where F is given by

$$F = s_k \frac{|u^{s_k}|^2}{k} + s_p \frac{|u^{s_p}|^2}{p} + s_q \frac{|u^{s_q}|^2}{q}. \quad (19)$$

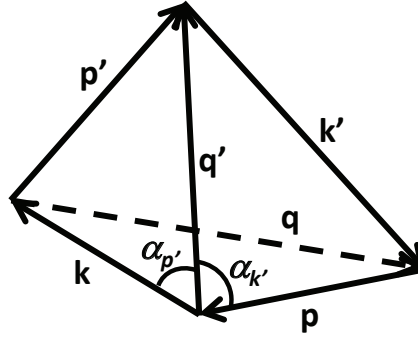


Figure 1. Tetrahedron configuration of interacting four triads $(\mathbf{k}, \mathbf{p}, \mathbf{q})$, $(\mathbf{k}, \mathbf{p}', \mathbf{q}')$, $(\mathbf{p}, \mathbf{q}', \mathbf{k}')$ and $(\mathbf{q}, \mathbf{k}', \mathbf{p}')$.

Defining a potential vector \mathbf{w} such that $\mathbf{u} = \nabla \times \mathbf{w}$, we have $\mathbf{w}(\mathbf{k}) = k^{-1} (u^+(\mathbf{k}) \mathbf{h}^+(\mathbf{k}) - u^-(\mathbf{k}) \mathbf{h}^-(\mathbf{k}))$, one can express a potential helicity in the form $F(\mathbf{k}) = \mathbf{u} \cdot \mathbf{w} = k^{-1} (|u^+(\mathbf{k})|^2 - |u^-(\mathbf{k})|^2)$. Therefore F above is the potential helicity for the whole triad $\Delta_{\mathbf{k}\mathbf{p}\mathbf{q}}$. We find that if $s_k = s_p = s_q$ then a is negative, suggesting that within a triad of only one type of helical modes it cannot lead to dynamo action. Taking $u^{s_k} = 0$ and $s_k = -s_p = -s_q$ leads to positive a . Then there is always a magnetic field scale k with helicity opposite to u such that dynamo action occurs. The importance of such potential helicity has already been put forward in the context of large scale dynamos [12] with indeed opposite helical modes between the large scale magnetic field and the flow.

3.2. Full triad

Considering all possible kinetic and magnetic helical modes within one triad b_k^\pm , u_k^\pm , b_p^\pm , u_p^\pm , b_q^\pm and u_q^\pm we find

$$a = -\frac{Q_{kpq}^2}{4} \left(\left| \frac{u_k^+ - u_k^-}{k} \right|^2 + \left| \frac{u_p^+ - u_p^-}{p} \right|^2 + \left| \frac{u_q^+ - u_q^-}{q} \right|^2 \right), \quad (20)$$

provided we choose $\mathbf{z}_k = \mathbf{z}_p = \mathbf{z}_q = \lambda$. Contrary to the previous cases a is always negative. It shows that one triad alone cannot lead to dynamo action (unless some helical magnetic modes are suppressed as previously seen). This result is consistent with the antidynamo theorem that says that no dynamo effect is possible for a magnetic field which depends only on two coordinates. Indeed for any triad $\Delta_{\mathbf{k}\mathbf{p}\mathbf{q}}$ there is always a rotational transformation which makes the problem two dimensional. So it is necessary to involve an additional magnetic field mode which does not belong to the plane $\Delta_{\mathbf{k}\mathbf{p}\mathbf{q}}$.

4. Solution in one tetrahedron

In addition to vectors \mathbf{k} , \mathbf{p} , \mathbf{q} we introduce an arbitrary wave number vector \mathbf{q}' . Then the tetrahedron is formed by four triads: $(\mathbf{k}, \mathbf{p}, \mathbf{q})$, $(\mathbf{k}, \mathbf{p}', \mathbf{q}')$, $(\mathbf{p}, \mathbf{q}', \mathbf{k}')$ and $(\mathbf{q}, \mathbf{k}', \mathbf{p}')$ as shown in figure 1. Vectors \mathbf{k}' and \mathbf{p}' have to be included for consistency.

We assume that the flow is given by the two vectors u_k and u_p only ($u_q^\pm = u_{q'}^\pm = u_{p'}^\pm = u_{k'}^\pm = 0$) and we solve the eigenvalue problem with all twelve magnetic field unknowns. We find a solution in the form

$$a = -\frac{Q_{kpq}^2}{4} \left| \frac{u_k^+ - u_k^-}{k} \right|^2 - \frac{Q_{pqk'}^2}{4} \left| \frac{u_p^+ - u_p^-}{p} \right|^2 \pm \frac{Q_{kp'q'} Q_{pqk'}}{2 q'} |\sin \psi_{q'}| \sqrt{\left(\frac{|u_k^+|^2 - |u_k^-|^2}{k} \right) \left(\frac{|u_p^+|^2 - |u_p^-|^2}{p} \right)} \quad (21)$$

where $\psi_{q'}$ is the angle between the planes defined by the triads $(\mathbf{k}, \mathbf{p}', \mathbf{q}')$ and $(\mathbf{p}, \mathbf{q}', \mathbf{k}')$. If the two triads are coplanar, corresponding to $\sin \psi_{q'} = 0$, then a is always negative, ruling out any possibility for dynamo action. If the flow is non helical ($u_k^+ = u_k^-$ and $u_p^+ = u_p^-$) then $a = 0$. Now if the two triads are non coplanar and the flow is helical then dynamo becomes possible. Oscillatory solutions are always possible provided $H(k) \cdot H(p) < 0$. If $H(k) \cdot H(p) > 0$ then only non local dynamo action is possible ($q' \ll k$ and $q' \ll p$).

5. Conclusion remarks

From solution (21) we can learn several things that are in favour of dynamo and provide highest growth rate: flow with maximum helicity, interaction of large magnetic field and small scale velocity field, orthogonal configuration of $\mathbf{u}(\mathbf{k})$, $\mathbf{u}(\mathbf{p})$ and $\mathbf{b}(\mathbf{q}')$. These are known since G.O. Roberts dynamo example [6]. He considered the flow $\mathbf{u} = (\sin y, \sin x, \cos x - \cos y)$ and found asymptotic unstable solution for $\mathbf{b}(x, y, z) = \mathbf{b}_0 e^{ik_z z}$ for $k_z < 1$. If we consider this solution in terms of helical modes then $\mathbf{k} = (1, 0, 0)$, $\mathbf{p} = (0, 1, 0)$ and $\mathbf{q}' = (0, 0, k_z)$. Also $u_k^+ = 1/2$, $u_k^- = 0$ and $u_p^+ = -1/2$, $u_p^- = 0$. Extracted highlights from (21) are in full agreement. We show that the simplest modes configuration leading to an unstable solution has the form of a tetrahedron. Here the dynamo is produced by only two scales flow. We find necessary conditions for the dynamo action which is expressed by helicities of the individual helical modes and not directly related to flow helicity. For the case of one triad with single helical modes (section 3.1) we have found the growth rate depending on potential helicity [12]. Our general approach is similar the study of low-dimensional magnetohydrodynamic models containing three velocity and three magnetic modes [13]. These low-dimensional models exhibit a dynamo transition at a critical forcing amplitude that depends on the Prandtl number. In comparison we took an advantage of helical mode decomposition that prevents break of conservation laws automatically.

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