Statistical Degradation Models for Reliability Analysis in Non-Destructive Testing

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Abstract. In this paper, we consider the application of the statistical degradation models for reliability analysis in non-destructive testing. Such models enable to estimate the reliability function (the dependence of non-failure probability on time) for the fixed critical level using the information of the degradation paths of tested items. The most widely used models are the gamma and Wiener degradation models, in which the gamma or normal distributions are assumed as the distribution of degradation increments, respectively. Using the computer simulation technique, we have analysed the accuracy of the reliability estimates, obtained for considered models. The number of increments can be enlarged by increasing the sample size (the number of tested items) or by increasing the frequency of measuring degradation. It has been shown, that the sample size has a greater influence on the accuracy of the reliability estimates in comparison with the measuring frequency. Moreover, it has been shown that another important factor, influencing the accuracy of reliability estimation, is the duration of observing degradation process.

1. Introduction

Usually, the estimates of reliability are constructed on the basis of the information about failures. However, in non-destructive testing, the researcher doesn't have any failure time data. Instead of that, there is the information about aging of tested items (the so-called degradation data). In this case, the value of some parameter (index), characterizing the process of degradation (aging) of tested items, is measured. The time moment, when the value of degradation index reaches the critical level, is called the failure time.

Different types of degradation models were considered in Meeker (1998) and Liao and Tseng (2006). The gamma degradation model, which is used when the increments of degradation index have the gamma distribution with shape parameter equal to the difference between the values of trend function at the corresponding moments of time, was considered in Bagdonavičius and Nikulin (2002). The goodness-of-fit test for the gamma degradation model was proposed in Chimitova and Chetvertakova (2014). Tsai et al. (2012) and Chimitova and Chetvertakova (2015) have studied the gamma model with random effects, which describes the unit-to-unit variability. However, this degradation model is suitable for the cases, when the increments of the degradation index are non-negative values. In other cases it is proposed to use the Wiener degradation model, which was considered, for example, by Whitmore (1995) and Tsai et al. (2011). This model includes the normal distribution as the distribution of degradation index increments, where the values of trend function are included in the shift and scale parameters. In addition, one of the advantages of using the gamma and

Wiener degradation models is that reliability estimates can be obtained analytically due to the repeatability of the normal distribution, as well as the gamma degradation model.

In this paper, we propose the usage of the gamma and Wiener degradation models for reliability analysis in nondestructive testing. By means of computer simulation technique, we analyze the accuracy of reliability function estimation. The recommendations for increasing the accuracy by changing the number of items or frequency of measuring the degradation index are given for linear and power trend functions.

2. Gamma Degradation Model

Stochastic process $Z(t)$ characterizing degradation process is referred to as the gamma degradation process, if

- $Z(0) = 0$;
- $Z(t)$ is a stochastic process with independent increments;
- increments $\Delta Z(t) = Z(t + \Delta t) - Z(t)$ have the gamma distribution with probability density function

$$f_{Gamma}\left(t; \sigma, \Delta v(t)\right) = \left(\frac{t}{\sigma}\right)^{\Delta v(t)-1} \frac{e^{-t/\sigma}}{\sigma \cdot \Gamma\left(\Delta v(t)\right)},$$

where $\Delta v(t) = v(t + \Delta t) - v(t)$ is the shape parameter and σ is the scale parameter, $v(t)$ is a positive increasing function.

If random variables ξ_1 and ξ_2 follow the gamma distribution with scale parameter σ and shape parameters v_1 and v_2, correspondingly, then $\xi_1 + \xi_2$ follows the gamma distribution with scale parameter σ and shape parameter $v_1 + v_2$. This property explains the fact of using the gamma distribution as a distribution of increments.

Let the mathematical expectation of degradation process $Z(t)$ is

$$M\left(Z(t)\right) = m(t),$$

where $m(t) = \sigma \cdot v(t)$ is a trend function of the degradation index.

In this paper, we consider two types of trend functions:

- linear trend function $m(t; \overline{\gamma}) = \gamma_0 + \gamma_1 t$;
- power trend function $m(t; \overline{\gamma}) = \gamma_0 t^{\gamma_1}$.

Taking into account the given assumptions, the stochastic process $Z(t)$ at time moment $t = t_k$ has the gamma distribution with the shape parameter equal to $\dfrac{m(t_k; \overline{\gamma})}{\sigma}$.

The time to failure is the random variable

$$T = \sup\{t : Z(t) < z_0\},$$

where z_0 is the critical value of the degradation path. Then, the reliability function for the gamma degradation model is given by:

$$S(t) = P\{T > t\} = P\{Z(t) < z_0\} = F_{Gamma}\left(z_0; \sigma, \frac{m(t; \overline{\gamma})}{\sigma}\right). \tag{1}$$

Let the realization of degradation path $Z^i(t)$ for the i-th item is denoted as

$$Z^i = \left\{(0, Z_0^i = 0), (t_1^i, Z_1^i), ..., (t_{k_i}^i, Z_{k_i}^i)\right\}, \ i = \overline{1, n},$$

where k_i is the number time moments, in which the degradation index was measured. Then, the sample of independent degradation index increments can be written as:

$$\mathbf{X}_n = \left\{ X_j^i = Z_j^i - Z_{j-1}^i, \, i = \overline{1,n}, \, j = \overline{1,k_i} \right\}. \tag{2}$$

Maximum likelihood estimates of parameters σ and $\overline{\gamma}$ are calculated by maximization of the log-likelihood function:

$$\ln L(\mathbf{X}_n) = \sum_{i=1}^{n} \sum_{j=1}^{k_i} \ln f_{Gamma}(X_j^i; \sigma, p_j^i) \to \max_{\sigma, \overline{\gamma}},$$

where $p_j^i = \dfrac{m(t_j^i; \overline{\gamma}) - m(t_{j-1}^i; \overline{\gamma})}{\sigma}$ is the shape parameter of the gamma distribution.

3. Wiener Degradation Model

Stochastic process $Z(t)$ characterizing degradation process is referred to as the Wiener degradation process, if ● $Z(0) = 0$; ● $Z(t)$ is a stochastic process with independent increments; ● increments $\Delta Z(t) = Z(t + \Delta t) - Z(t)$ have the normal distribution with probability density function

$$f_{Norm}(t; \theta_1, \theta_2) = \frac{1}{\sqrt{2\pi} \cdot \theta_2} e^{-\frac{1}{2}\left(\frac{t - \theta_1}{\theta_2}\right)^2},$$

where $\theta_1 = m(t + \Delta t) - m(t)$ is the shift parameter and $\theta_2 = \sigma \sqrt{m(t + \Delta t) - m(t)}$ is the scale parameter, $m(t)$ is a positive increasing trend function:

$$M\big(Z(t)\big) = m(t).$$

It is well known, that if random variables ξ_1 and ξ_2 follow the normal distribution with shift parameters μ_1 and μ_2 and scale parameters σ_1 and σ_2, correspondingly, then $\xi_1 + \xi_2$ follows the normal distribution with shift parameter $\mu_1 + \mu_2$ and scale parameter $\sqrt{\sigma_1^2 + \sigma_2^2}$.

Then, taking into account the given assumptions, the stochastic process $Z(t)$ at time moment $t = t_k$ has the normal distribution with the shift parameter equal to $m(t_k; \overline{\gamma})$, and the scale parameter equal to $\sigma \sqrt{m(t_k; \overline{\gamma})}$. In this case, the reliability function for the Wiener degradation model is given by:

$$S(t) = P\{T > t\} = P\{Z(t) < z_0\} = \Phi\left(\frac{z_0 - m(t; \overline{\gamma})}{\sigma \sqrt{m(t; \overline{\gamma})}}\right), \tag{3}$$

where $\Phi(\cdot)$ is the standard normal distribution function.

The log-likelihood function for this model can be written as:

$$\ln L(\mathbf{X}_n) = \sum_{i=1}^{n} \sum_{j=1}^{k_i} \ln f_{Norm}\left(X_j^i; m(t_j^i; \overline{\gamma}) - m(t_{j-1}^i; \overline{\gamma}), \sigma \sqrt{m(t_j^i; \overline{\gamma}) - m(t_{j-1}^i; \overline{\gamma})} \right),$$

where $\mathbf{X}_n$ is the sample of independent degradation index increments (2).

4. Investigation of the Reliability Estimates in Degradation Analysis

It is clear, that the accuracy of estimates of reliability functions, obtained on the basis of degradation data, strongly depends on the number of degradation increments. So, it is possible to improve the estimation accuracy by increasing the number of items n and/or the frequency of measuring the degradation index k_i. In this section, we investigate the accuracy of estimation of model parameters and reliability functions (1) and (3) by means of computer simulation technique.

The accuracy of parameters estimation was measured with the value

$$g_N(\theta) = \frac{1}{N} \sum_{i=1}^{N} \left(\hat{\theta}_i - \theta_{true} \right)^2,$$

where N is the number of simulated samples, $\hat{\theta}_i$ is the maximum likelihood estimate of parameter θ by the i-th sample, θ_{true} is the true value of parameter.

The value d_N was used as the measure of accuracy for the reliability function estimates:

$$d_N = \frac{1}{N}\sum_{i=1}^{N}\sup_{t\geq 0}\left|S_0(t)-\hat{S}_i(t)\right|,$$

where $\hat{S}_i(t)$ is the reliability function obtained after parameters estimation by the i-th sample and $S_0(t)$ is the true reliability function.

In Tables 01 and 02 the values of $g_N(\theta)$ and d_N are given for the gamma and Wiener degradation models, correspondingly. We have taken the following true values of parameters:

- $\sigma = 0.5$, $\gamma_0 = 0$ (was not estimated), $\gamma_1 = 0.5$ in the case of linear trend and
- $\sigma = 0.5$, $\gamma_0 = 1$, $\gamma_1 = 1.3$ in the case of power trend function.

The critical degradation value $z_0 = 10$ was taken for both gamma and Wiener models. The number of items $n = 10, 20$ and the number of time moments for each item $k = 10, 20$. The time moments were taken from 0 to 19 with the step $\Delta t = 2$ for $k = 10$ and $\Delta t = 1$ for $k = 20$.

The number of simulations $N = 1000$.

Table 1. The accuracy of estimation of model parameters and reliability function (1)

		Linear trend			Power trend			
n	k	d_N	$g_N(\gamma_1)$	$g_N(\sigma)$	d_N	$g_N(\gamma_0)$	$g_N(\gamma_1)$	$g_N(\sigma)$
10	10	0.3626	0.0063	0.0359	0.2147	0.0199	0.0020	0.0198
20	10	0.2504	0.0030	0.0144	0.1218	0.0144	0.0012	0.0086
10	20	0.318	0.0061	0.0182	0.1693	0.0178	0.0018	0.0091

In Figure 1, the reliability functions $\hat{S}_i(t)$, $i = \overline{1,50}$, obtained for the gamma degradation model with linear trend functions and $n = 20, k = 10$ (grey lines), as well as the true reliability function (black line) are presented.
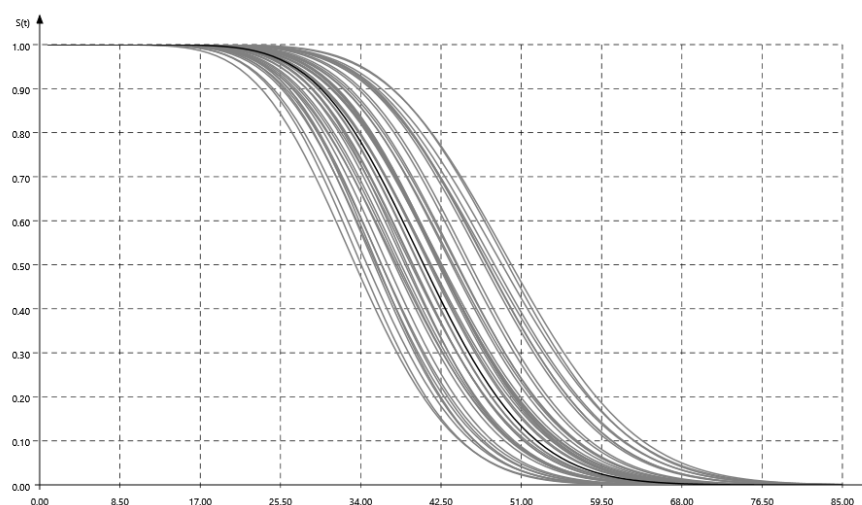

Figure 1. Reliability functions for the gamma degradation model with the linear trend for $n = 20$, $k = 10$, $z_0 = 10$.

Table 2. The accuracy of estimation of model parameters and reliability function (3)

		Linear trend			Power trend			
n	k	d_N	$g_N(\gamma_1)$	$g_N(\sigma)$	d_N	$g_N(\gamma_0)$	$g_N(\gamma_1)$	$g_N(\sigma)$
10	10	0.1161	0.0950	0.0092	0.0898	0.0262	0.0031	0.0070
20	10	0.0888	0.0912	0.0028	0.0673	0.0074	0.0008	0.0035
10	20	0.1124	0.089	0.0038	0.2831	0.0198	0.0022	0.0024

As can be seen from Tables 1 and 2, the estimates, which were obtained in the case of $n=20, k=10$, are more accurate for both trend functions and degradation models. It means, that it is more preferable to improve the estimation accuracy by increasing the number of tested items instead of increasing the frequency of measuring.

The accuracy of reliability estimation depends not only on the sample size and measuring frequency. A very important factor is the amount of information about lifetime in observed degradation paths. In Figures 02 and 03, there are the examples of degradation paths with critical levels $z_0=10$ and 20, correspondingly. It is obvious, that in the case of $z_0=10$ some items have reached the critical level, so the researcher has more information about lifetimes, in contrast to the situation, when the critical level is equal to 20.
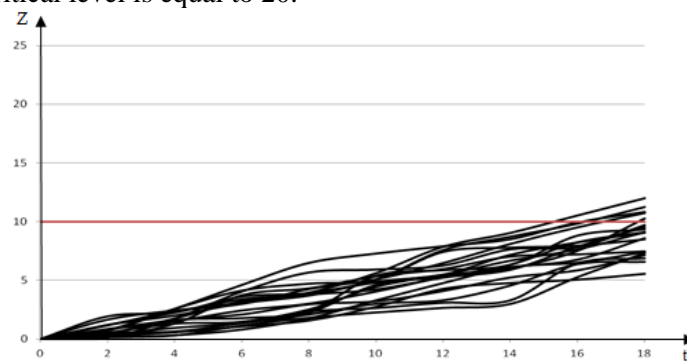

Figure 2. Degradation paths and critical level $z_0=10$.

In Figure 4, there are the reliability functions $\hat{S}_i(t)$, $i=\overline{1,50}$, obtained for the same gamma degradation model as in Figure 1, but for $z_0=20$. Comparing Figures 1 and 4, one can see, that the reliability estimates are much more accurate in the case of $z_0=10$.
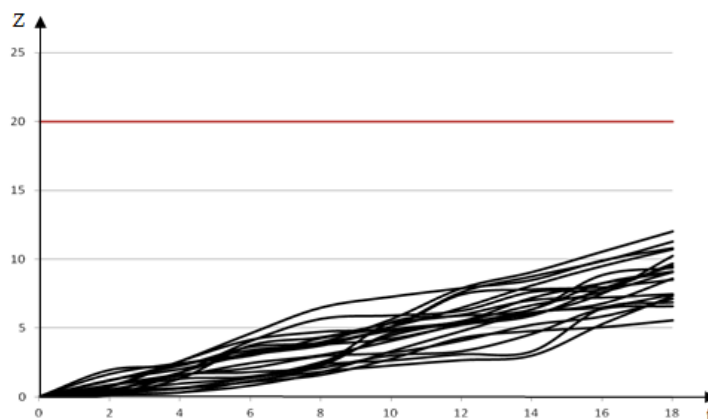

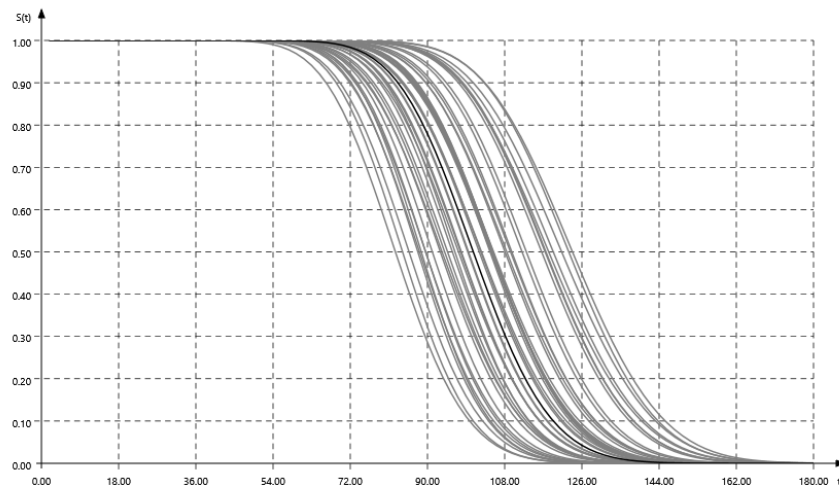**Figure 3.** Degradation paths and critical level $z_0=20$.

Figure 4. Reliability functions for gamma degradation model with the linear trend and $n = 20$, $k = 10$, $z_0 = 20$.

5. Summary
In this paper, we have analyzed the possibility of reliability estimation in nondestructive testing on the basis of degradation processes. Two widely used degradation models: the gamma and Wiener models have been considered. Summarizing the investigation results, we can conclude that the sample size has a greater influence on the accuracy of the reliability estimates in comparison with the frequency of measuring degradation. Moreover, it has been shown that the more complete degradation paths were observed (the closer degradation index to the critical level), the more accurate reliability estimates will be obtained.

Acknowledgments

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