Cryogenic parallel, single phase flows: an analytical approach

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Abstract. Managing the cryogenic flows inside a state-of-the-art accelerator cryomodule has become a demanding endeavour: In order to build highly efficient modules, all heat transfers are usually intercepted at various temperatures. For a multi-cavity module, operated at 1.8 K, this requires intercepts at 4 K and at 80 K at different locations with sometimes strongly varying heat loads which for simplicity reasons are operated in parallel. This contribution will describe an analytical approach, based on optimization theories.

1. Introduction

The design of efficient cryo-modules usually demands cooling of many intercepts, and even the heat loads itself are sometimes at different temperatures. In the Cornell ERL main linac cryo-module (MLC) for example, the main heat load is at 1.8 K – which are the SRF cavities[1]. Nevertheless, there is also an 80 K heat load being the higher order mode (HOM) absorbers. Besides that, intercept cooling is provided for all couplers at 80 K and at 5 K, and a 5 K cooling is also provided to the beamline flanges at the HOM absorber. Similar cooling requirements can be found at the LCLS-II cryomodule[2].

During the design phase of the cryogenic cooling arrangement an important design choice has to be made: Do these heat loads have to be intercepted by an active cooling channel with a coolant stream or is a passive cooling provided by straps sufficient?

For smaller heat loads a passive cooling is usually chosen. In this arrangement, the coolant flows through a rather large channel while the intercepts are cooled passively by straps and heat conduction. As a matter of fact, temperatures at the intercept can be significantly higher than the temperature at the coolant. As an example, the LCLS-II cryo-module power coupler is cooled passively. With an expected heat load of 20 W the intercept will operate at 120 K even though the cooling is provided by an 80 K gas stream, indicating a temperature gradient along the cooling strap of 40 K [3].

Above certain heat loads, conductive cooling becomes unfeasible (in terms of having a short path and a large cross-sections). For the Cornell injector cryomodule (ICM), where the expected heat load on the power couplers was 75 W at 80 K, active cooling was chosen. In this arrangement, the coolant flows directly to the intercept guaranteeing a minimum temperature gradient[4].

If cooling is required at several intercept, the question arises whether to cool them in series or in parallel. A serial cooling is easy to design but as a consequence of the individual heating applied, temperature of the intercepts will increase along the flow path. In contrast, parallel cooling allows uniform temperatures at all intercepts but designing the manifold such that every stream gets an appropriate mass flow is quite demanding for several reasons. Firstly, pipe diameters and lengths have to been chosen carefully and it has to made sure that a change in one cooling channel is correctly

compensated by and adequate change in the parallel flow. Secondly, parallel flows are highly susceptible to become unbalanced. This phenomena, commonly know as "the coldest stream starves the warmers" is a result of the fact that the effective fluid impedance of a channel is a function of the fluid parameters itself and higher temperatures of the fluid lead to a higher impedance of a channel. This effect becomes an issue, if heat loads are not well known – like in a prototype- or if the heat loads are of statistical character – like the heating from higher order modes.

For these reasons, there seems to be a great reluctance to use parallel flows for intercept cooling. At Cornell, throughout the design of several cryomodule with parallel flows, we developed a tool-box of design aids, starting from simple spread sheet to dynamics simulators that allow simulation of transients [5,6]. Part of that effort was driven by the fact that the 80 K heat loads of our ICM where quite different from the expected, some design changes in the HOM absorbers cooling where not compensated by accompanying modification in the parallel circuit.

This paper introduces a different approach: instead of calculating the fluid dynamics step-wise using database values for the coolant properties I will describe an approach based on the mathematical theory to optimization, developed by Lagrange as well as Kuhn and Tucker. I will show that this approach will result in an analytical solution of the parallel flow characteristics.

2. Background

Let's assume a simplified parallel flow circuit as given in figure 1. Let it consist of two parallel streams of Helium ($i = 1,2$) with two independent heat loads Q_i on each branch. Let's assume the instream of the helium is at a given temperature T_{in} and pressure p_{in} that the mass-flow is a controlled and known quantity $\dot{m}_{tot}$. The dimensions of each flow channel is given its length L_i and its hydraulic diameter D_i.

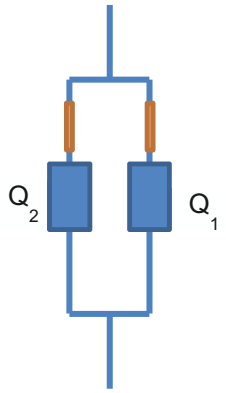

Figure 1. Simplified model of a parallel flow circuit

As a result of the coolant flowing through the channels there will be a decrease in the pressure which is given by the Darcy-Weisbach equation

$$\Delta p_i = \frac{f_i L_i \rho_i v_i^2}{2 D_i} \tag{1}$$

which not only includes the properties of the geometry but also quantities characterizing the fluid like the density ρ_i and the friction factor f (being eventually correlated with the Reynolds number) definitions of which are given in text books (for example [7] or [8]). Some parameters are given by the mechanical layout (like D_i and L_i), while others are fluid properties (ρ, μ) and are usually taken from databases like HEPAK [7], based in the inlet parameters p_{in} and T_{in}. For complex geometries or in situations where T (because of the heat load Q) and p (as a result of the pressure drop from the preceding piping) are not constant, pressure drops have to be calculated stepwise following eqn. (1).

Eventually, the mass flows $\dot{m}$ and thus the fluid velocities v are a result of the condition that the pressure drop along each parallel path has to be equal. In our case the guiding condition is

$$\Delta p_1 = \Delta p_2. \tag{2}$$

For the calculations described [5] and [6] complex tables where used and the condition of eqn. (4) was fulfilled by adjusting the massflow $\dot{m}$ manually (as done in [5]) or by using scripting and seek function techniques as done in [6]. Even though solutions have been found, the method is basically a numerical iteration. However, eqn. 2 can be transformed into a minimization problem $min\{(\Delta p_1 - \Delta p_2)^2\}$, which seem to be a strange approach, but allows an analytical solution using well know optimization conditions described in the section below.

3. An analytical approach based on optimization theories

3.1. Motivation

In classical mechanical system the dynamics is well described by Newton's laws of motion. Applying these laws under boundary conditions, for example on a roller coaster, where the motion is constrained by the rails however can be quite demanding. In these cases, the Lagrangian approach to the classical motion has found to be more adequate. Here, the equation of motion can are derived by determining the Lagrangian multipliers that correspond to the optimization problem $max\{T - V\}$, where T is the kinetic and V the potential energy of the system.

3.2. Lagrange Form

Analogue to the mechanical example, flow rates of the parallel channels are such that the pressure decrease in both channels are equal. This transforms into this optimization problem

$$max\{-(\Delta p_1 - \Delta p_2)^2\} \tag{3}$$

under the boundary conditions of

$$m_1 + m_2 = m_{tot}. \tag{4}$$

Additional constraints might be given (maybe Q_i to have a certain value). The Lagrangian function is then given by:

$$\mathcal{L} = -(\Delta p_1 - \Delta p_2)^2 \ - \lambda(m_1 + m_2 - m_{tot}). \tag{5}$$

According to the Lagrangian theory, the optimum solution can be derived by solving this set of equations: $\nabla\mathcal{L} = 0$, which explicitly written is

$$\frac{d\mathcal{L}}{dm_1} = 0, \frac{d\mathcal{L}}{dm_2} = 0, \frac{d\mathcal{L}}{d\lambda} = 0 \tag{6}$$

This set of 3 equations allows calculating the fluid dynamics with 3 free parameters. Sometimes, more than 3 free parameters exists which means the designer has to make a reasonable choice, or add more conditions to constrain the problem.

So far, the optimization uses boundary conditions which have to be exactly fulfilled- but this limitation can be overcome by the method described next.

3.3. Kuhn-Tucker condition

If the boundary conditions are more complex, the problem becomes a nonlinear optimization, which can be handled by the Kuhn-Tucker conditions. Let's assume we still want to maximize

$$-(\Delta p_1 - \Delta p_2)^2 \stackrel{\text{def}}{=} F(\Delta p_1, \Delta p_2), \tag{7}$$

which we will call function F for convenience reasons. In contrast to the Lagrangian situation, the boundary conditions are now non-linear or of relation type. A typical design constrain is for example that the overall mass is not to exceed a certain value, or that the temperature increase at an heat load intercept has to be smaller than an allowable figure:

$$m_1 + m_2 \leq m_{tot}, \Delta T_1 \leq 20K,\ \Delta T_2 \leq 50K \tag{8}$$

On an abstract level, the boundary conditions are described by a set of relations given by $g_i < c_i$. The Lagrangian function is then given by

$$\mathcal{L} = -(\Delta p_1 - \Delta p_2)^2\ - \lambda_1(m_1 + m_2 - m_{tot}) -\ \lambda_2(\Delta T_1 - 20K) - \lambda_3(\Delta T_2 - 50K). \tag{9}$$

In this case now, the optimum solution is not only given by the Lagrangian multipliers (which would only be true if all boundary conditions are binding), but would also solve the three Kuhn-Tucker conditions. The first Kuhn-Tucker condition is described by this set of equations, where the number of equations is equal to the free parameters in eqn. 7:

$$\begin{aligned} \frac{\partial f}{\partial \Delta p_1} - \lambda_1 \frac{\partial g_1}{\partial \Delta p_1} \ldots \lambda_3 \frac{\partial g_3}{\partial \Delta p_1} = 0 \\ \frac{\partial f}{\partial \Delta p_2} - \lambda_1 \frac{\partial g_1}{\partial \Delta p_2} \ldots \lambda_3 \frac{\partial g_3}{\partial \Delta p_2} = 0 \end{aligned} \tag{10}$$

The second Kuhn-Tucker condition ensures that the Lagrangien multipliers gained from eqn. 9 describe a valid solution:

$$\lambda_1 \geq 0, \lambda_2 \geq 0\ and\ \lambda_3 \geq 0 \tag{11}$$

While in Lagrangian problems all multipliers λ_i are greater than 0, boundary conditions of the type described by eqn. (8) can lead to multipliers being 0.

Eventually, the third Kuhn-Tucker condition, described by

$$g_1 \leq c_1, g_2 \leq c_2 \quad and\ g_3 \leq c_3 \quad \geq 0, \tag{12}$$

allows the distinction between binding boundary conditions and non-binding conditions. If $g_i = c_i$, the condition is binding which means it restricts the optimization. Increasing c_i can than lead to a better solution. In the case of $g_i < c_i$ the boundary condition is called non binding.

It should be noted, that a mixed problem, defined by boundary conditions of the type

$$g_i \leq c_i, h_j = k_j \tag{13}$$

with c_i being design parameter given as an upper limit and k_j being parameters to be met exactly, can also be well calculated using the Kuhn-Tucker conditions. For parameters with an upper and a lower limit (like $\dot{m}_{1,l} \leq \dot{m}_1 \leq \dot{m}_{1,u}$), two boundary conditions have to be modelled.

4. Applying nonlinear optimization to fluid dynamics

The optimization theory described can be applied to fluid dynamics calculations in may ways:

- Calculating the mass flow ratio through parallel flows (like done above)
- Optimize the overall mass-flow through a system
- Calculating parameters of a flow restrictor to balance or stabilize parallel flows
- Optimize a system with uncertain heat loads and
- More sophisticated parameter studies.

However, setting up the problem can be tricky. For example, the pressure drop according to eqn. 1 is a complicated function of many parameters like the density, viscosity, temperature and the pressure of the fluid, the flow regime, heat loads and details of the geometry of the piping. Some formulas are given above, others are omitted in this context but can easily be found in text book.

When translating the fluid flow problem into an optimization problem, material and fluid parameters, usually taken from a database, have to be parameterized. For example, the density of helium at 80 K is pretty accurately described by the ideal gas law. At 5 K, modifications have to be applied, however, low polynomial approximations exist. For example the dynamic viscosity is well described by an experimentally gained formula $\mu = 5.18 \cdot 10^{-7} \cdot T^{0.64}\ [Pa \cdot s]$. Usually older publications are full of those approximation relations.

The more demanding portion in setting up the problem is to translate eqn (1) such that it is only a function of the optimization parameters. If, for example, the geometry is given, L_i and D_i are known. In a typical scenario, T_{in} and p_{in} would also be given and we assume you want to optimize the mass flows m_i for various assumptions of Q_i. Then, eqn (1) has to be a sole function of m_i. In the case mentioned, this becomes

$$\Delta \mathrm{p} = 0.156 \cdot \mathrm{D}^{-4.8} \cdot \rho^{-1} \cdot \mu^{-0.2} \cdot \mathrm{L} \cdot \dot{\mathrm{m}}^{2.2}, \tag{14}$$

so the function to maximize will be

$$\begin{aligned} \mathrm{F}(\dot{m}_1, \dot{m}_2) = -\big(& D_1^{-9.6} \cdot \rho_1^{-2}(p_{in}, T_{in}) \cdot \mu_1^{-0.4} \cdot L_1^2 \cdot \dot{m}_1^{4.4} + D_2^{-9.6} \cdot \rho_2^{-2}(p_{in}, T_{in}) \cdot \mu_2^{-0.4} \cdot L_2^2 \\ & \cdot \dot{m}_2^{4.4} - 2\, D_1^{-4.8} D_2^{-4.8} \cdot \mu_1^{-0.2} \mu_2^{-0.2} \cdot L_1 \cdot L_2 \cdot \rho_1^{-1}(p_{in}, T_{in}) \\ & \cdot \rho_2^{-1}(p_{in}, T_{in}) \cdot \dot{m}_1^{2.2} \cdot \dot{m}_2^{2.2}\big). \end{aligned} \tag{15}$$

Just as demonstration, one may calculate the analytical optimization of a simple parallel flow (as depicted in fig.1) where the heat loads are equal on each branch, but the pipe length L_1 and L_2 are different and a certain overall massflow $\dot{m}_{tot}$is given. Using eqn (1) a Lagrangian function according to eqn (5) can be set-up. Solving the optimization problem (eqn (3)) requires solving the system of equations given by eqns (6), which leads eventually to a quadratic problem, the solution of which is given by:

$$\begin{aligned} L_1 = \frac{1}{2} L_2 \Bigg[& \left(\frac{\dot{m}_{tot} - \dot{m}_1}{\dot{m}_1}\right)^{0.8} - \left(\frac{\dot{m}_1}{\dot{m}_{tot} - \dot{m}_1}\right)^{0.2} \\ & + \sqrt{\left(\frac{\dot{m}_{tot} - \dot{m}_1}{\dot{m}_1}\right)^{1.6} + 2\left(\frac{\dot{m}_{tot} - \dot{m}_1}{\dot{m}_1}\right)^{0.6} - 4\left(\frac{\dot{m}_1}{\dot{m}_{tot} - \dot{m}_1}\right)^{0.4}} \Bigg] \end{aligned} \tag{16}$$

The result itself might not be very exciting: depending on what mass flow you choose for path 1, the length of that piping has to follow eqn (16) for a given length of the parallel pipe L_2. In fact

plugging in numbers for the case describe in [5] leads to the exact same results. In that sense, using the optimization approach is not superior to the numerical approach we have chosen in the past.

However, it need to be mentioned, that eqn (16), as well as any other solution of the optimization problem, is still an analytical expression. That means, one can easily investigate sensitivities to parameter changes or even do large parameter studies. This is especially valuable if for example heat loads are unknown or of statistical character (like in our initial problem set, where the heat loads from the higher order mode absorbers are of that type).

5. Conclusion

Mathematical optimization methods like the Lagrange optimization or the non-linear optimization using Kuhn-Tucker conditions can be an alternative to iterative methods of calculating fluid dynamics. Even though setting up the problem initially can be tricky, as well as solving the system of equations gained from eqn. (6) or especially from eqns. (10-12), the result will be a fully analytical solution of the problem. If appropriate propagation of a free parameter through all equations and the optimization calculation is ensured the analytical result will be a function of exactly this parameter, which then allows very efficient studies of primary and secondary dependencies.

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