A DRM for steady water absorption from different types of periodic channels

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Abstract. In this paper, problems involving steady infiltration from a homogeneous soil with water absorption by plant roots are studied. Different geometries of periodic channels are considered. The governing equation of the problems is a Richards equation. This equation may not be solved analytically. Hence, a numerical method is needed. To do so, the equation is transformed into a modified Helmholtz equation using a set of transformations involving Kirchhoff transformation, dimensionless variables and an exponential transformation. The modified Helmholtz equation is then solved numerically using a Dual Reciprocity Method (DRM) with a predictor-corrector scheme. Results obtained from channels with different geometries are presented.

1. Introduction

Problems involving water infiltration from channels into a homogeneous soil have been studied by numerous researchers, such as Solekhudin and Ang [6, 7, 8], and Clements *et al*. [2]. While Clements *et al*. studied infiltration from channels with impermeable layers [2], Solekhudin and Ang examined infiltration from periodic channels with different geometries [7]. Time-dependent infiltration from a circular channels has been considered by Clements and Lobo [3]. In these studies, water absorption by plant roots has not been taken into account. Hence, Solekhudin and Ang studied problems involving infiltration from periodic trapezoidal channels with water absorption by plant roots [6, 8]. However, in their studies Solekhudin and Ang did not consider different types of channels geometry. In this paper, we consider different types of channels geometry with water absorption by plant roots. The root distribution in this study is different from that in [6, 8].

2. Problem Formulation

Referred to a Cartesian coordinates $OXYZ$ consider an isotropic soil, Pima Clay Loam (PCL), in the region $Z \geq 0$ with OZ positively downward. Periodic irrigation channels are created on the surface of soil. The channels has surface area $2L$ per unit length the OY direction. The distance between the centre of two adjacent channels is $2(L+D)$. A row of crops are planted and equidistant between two adjacent channels. Water flows from the channels into the soil. An illustration of the description is shown in Figure 1. The root-water uptake used in this study is adopted from one of those reported in [10], which is Root A.
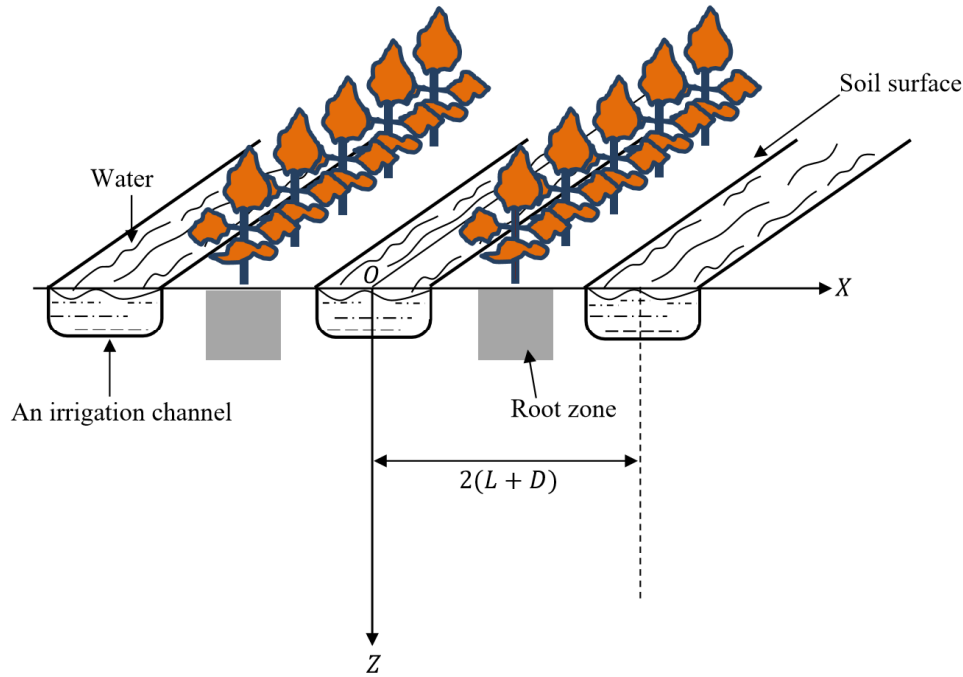
Figure 1. Periodic irrigation channels with rows of crops.

It is assumed that geometries of the channels and root zones do not alter in the *OY* direction. Hence, the problem is symmetry about the planes $X = \pm k(L+D)$, $k = 0, 1, 2, \cdots$. Due to the symmetry of the problem it is sufficient to consider a semi infinite region defined by $0 \leq X \leq L+D$ and $Z \geq 0$. This region is denoted by *R* bounded by *C*. The fluxes over the surface of channels are uniform, that is v_0, while over the surface of soil outside the channels are 0. There are no fluxes across $X = 0$ and $X = L+D$ as the problem symmetrical about them. Following Batu [1], we have $\partial\Theta/\partial X \to 0$ and $\partial\Theta/\partial Z \to 0$ as $X^2+Z^2 \to \infty$. Given this situation, we wish to determine suction potential and the water uptake from the three different geometries of periodic channels.

3. Basic Equations

In this study, equations and method of solution used are as discussed in Solekhudin and Ang [6]. However, for completeness, the equations and the method are briefly presented in this section. The governing equation of water infiltration with root-water uptake that may be used is

$$\frac{\partial}{\partial X}\left(K \frac{\partial \psi}{\partial X}\right) + \frac{\partial}{\partial Z}\left(K \frac{\partial \psi}{\partial Z}\right) - \frac{\partial K}{\partial X} = S(X,Z,\psi), \tag{1}$$

where *K* is the hydraulic conductivity, ψ is the suction potential and *S* is the root-water uptake function formulated as

$$S(X,Z,\psi) = \gamma(\psi)S_m(X,Z), \tag{2}$$

where

$$S_m(X,Z) = \frac{L_t\beta(X,Z)T_{pot}}{\displaystyle\int_0^{Z_m}\int_{L+D-X_m}^{L+D}\beta(X,Z)dXdZ}. \tag{3}$$

Here γ is the dimensionless water stress response function, L_t is the width of the soil surface associated with the transpiration rate, T_{pot} is the potential transpiration, and β is the spatial root-water uptake

distribution. The function γ used in this study is the same as that used in [8]. The spatial root-water uptake distribution is formulated as

$$\beta(X,Z) = \left(1 - \frac{L+D-X}{X_m}\right)\left(1 - \frac{Z}{Z_m}\right)e^{-\left(\frac{p_Z}{Z_m}|Z*-Z| + \frac{p_X}{X_m}|X*-(L+D-X)|\right)},$$ (4)

$$\text{for } L+D-X_m \leq X \leq L+D, \, 0 \leq Z \leq Z_m.$$

where X_m is half of the width of the root zone, Z_m is the depth of the root zone, and p_Z, p_X, $X*$, and $Z*$ are empirical parameters. The flux normal to the surface with outward pointing normal $\mathbf{n} = (n_1, n_2)$ is given by

$$F = U\, n_1 + V\, n_2,$$ (5)

where

$$U = -\frac{\partial\Theta}{\partial X} \text{ and } V = \alpha\Theta - \frac{\partial\Theta}{\partial Z}$$

Using the Kirchhoff transformation

$$\Theta = \int_{-\infty}^{\psi} K\,ds.$$ (6)

where Θ is the Matric Flux Potential (MFP), an exponential relationship between K and ψ,

$$K = K_s e^{\alpha\psi}, \, \alpha > 0$$ (7)

where K_s is the saturated hydraulic conductivity, the suction potential can be formulated as

$$\psi = \frac{1}{\alpha}\ln\left(\frac{\alpha\Theta}{K_s}\right).$$ (8)

Using Equations (6), (7) and substituting dimensionless variables

$$x = \frac{\alpha}{2}X; \quad z = \frac{\alpha}{2}Z; \quad \Phi = \frac{\pi\Theta}{v_0 L}; \quad u = \frac{2\pi}{v_0\alpha L}U; \quad v = \frac{2\pi}{v_0\alpha L}V; \quad f = \frac{2\pi}{v_0\alpha L}F;$$ (9)

into Equation (1) yield

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial z^2} - 2\frac{\partial\Phi}{\partial z} = \gamma*(\Phi)s*(x,z),$$ (10)

where

$$s*(x,z) = \frac{2\pi}{\alpha L}\frac{l_t\beta*(x,z)}{\displaystyle\int_0^{z_m}\int_{b-x_m}^{b}\beta*(x,z)\,dx\,dz}\frac{T_{pot}}{v_0}$$ (11)

and

$$\beta*(x,z) = \left(1 - \frac{b-x}{x_m}\right)\left(1 - \frac{z}{z_m}\right)e^{-\left(p_Z\left|Z*-\frac{2z}{\alpha}\right| + p_X\left|X*-\frac{2(b-x)}{\alpha}\right|\right)},$$ (12)

Here $l_t = \alpha L_t/2$, $x_m = \alpha X_m/2$, $z_m = \alpha Z_m/2$ and $b = \alpha(L+D)/2$. Applying transformation

$$\Phi = \phi e^z,$$ (13)

into Equation (10) yields

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} - \phi = \gamma*(\phi)s*(x,z)e^{-z}.$$ (14)

Equation (14) subject to boundary conditions

$$\frac{\partial\phi}{\partial n} = \frac{2\pi}{\alpha L}e^{-z} - n_2\phi \text{ on the surface of the channel,}$$ (15)

$$\frac{\partial\phi}{\partial n} = -\phi \text{ on the surface of soil outside the channel,}$$ (16)

$$\frac{\partial \phi}{\partial n} = 0, \quad x = 0 \text{ and } z \geq 0, \tag{17}$$

$$\frac{\partial \phi}{\partial n} = 0, \quad x = b \text{ and } z \geq 0, \tag{18}$$

$$\frac{\partial \phi}{\partial n} = -\phi, \quad 0 \leq x \leq b \text{ and } z = \infty. \tag{19}$$

An integral equation for solving Equation (14) is

$$\lambda(\xi,\eta) = \iint_R \varphi(x,z;\xi,\eta) \left[\phi(x,z) + \gamma*(\phi)s*(x,z)e^{-z} \right] dxdz$$

$$+ \int_C \left[\phi(x,z)\frac{\partial}{\partial n}\big(\varphi(x,z;\xi,\eta)\big) - \varphi(x,z;\xi,\eta)\frac{\partial}{\partial n}\big(\phi(x,z)\big) \right] ds(x,z), \tag{20}$$

where $\varphi(x,z;\xi,\eta) = (1/4\pi)\ln[(x-\xi)^2 + (y-\eta)^2]$ is the fundamental solution of Laplace equation, and

$$\lambda(\xi,\eta) = \begin{cases} 1, & \text{for } (\xi,\eta) \in R \\ 1/2, & \text{for } (\xi,\eta) \text{ on smooth part of } C \end{cases}$$

Equation (20) may be solved using a DRBEM and a predictor-corrector simultaneously. Readers may refer to [6] for the detail of the method.

4. Results and Discussion

The method described in the preceding section is tested on problems involving infiltration from three different geometries periodic channels into Pima Clay Loam (PCL) with root-water uptake process. The channels considered in this study are trapezoidal, semi-circular and triangular channels. We set $L = D = 50$ cm. The width of the three geometries of channels is the same, which is $4L/\pi$. For trapezoidal channels, the depth of the channels is $3L/2\pi$. It is assumed that the width and depth of the root zone are $2X_m = 100$ cm and $Z_m = 100$ cm, respectively. The potential transpiration rate, T_{pot}, is 4 cm/day, which was also used by Li et al [4], and Šimunek and Hopmans [5] in their studies. The values of experimental parameters α and K_s of the soil are 0.014 cm^{-1} and 9.9 cm/day, respectively [7]. The root-water stress response function γ used here is identical to that reported by Utset *et al.* [9], which can be seen graphically in Figure 2. The value of h_3 for $T_{pot} = 0.4$ cm/day is interpolated from $h_{3,a}$ and $h_{3,b}$, and we have $h_3 = 470$. Values of parameters appear in the root-water uptake function S are summarized in Table 1. The Parameter values in Table 1 are the same as those reported in [10].
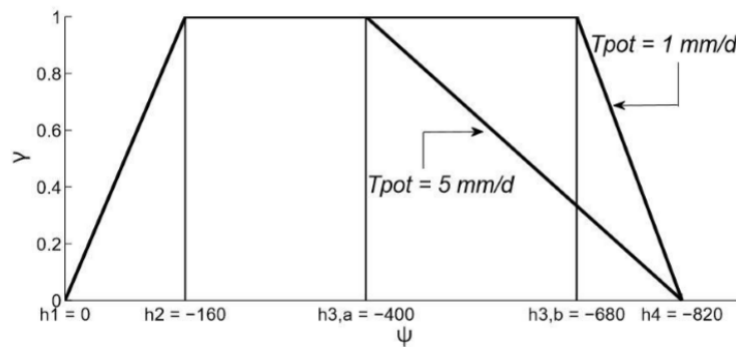

Figure 2. Graph of Root-water stress response function reported by Utset *et al.* [9]

Table 1. Parameters appear in the root-water uptake function.

P_Z	P_X	Z^*	X^*
1.00	1.00	0.00 cm	0.00 cm

The DRM with the predictor-corrector scheme is employed to obtain numerical solutions to equation (14). Using the numerical solutions, Transformation (13) and (9), numerical values of suction potential, ψ, can be obtained using Equation (8). Substituting ψ to equation (2) yield values of root uptake function, S. To employ the DRM, the domain must be bounded by a simple closed curve. The domain is set to be between $z = 0$ and $z = 4$, sufficient depth for boundary conditions to be applied without significant impact to values of Φ in the domain. To employ the method, the boundary must be discretized into a number of line segments, and a number of interior points are chosen as collocation points. The numbers of line segments and interior collocation points for three different geometries of channels are summarized in Table 2.

Table 2. Numbers of line segments and interior points chosen for three different geometries of channels.

	Triangular	Semicircular	Trapezoidal
Line Segments	200	201	202
Interior points	622	620	619

These numbers of line segments and interior points are chosen in such a way, such that an optimum computational time and the convergence of the values of Φ are achieved after several computational experiments. Some of the results are presented graphically in Figures 4 and 5. Figure 4 shows values of dimensionless MFP, Φ, for the three channels, at $x = 0.385$, $x = 0.525$ and $x = 0.665$. The three values of x are outside the channels. Specifically, Figure 3(a) shows the values of Φ for semicircular channels, Figures 3(b) and (c) show the values of Φ for trapezoidal and triangular channels, respectively. Dashed lines are the values of Φ for infiltration without root-water uptake. Solid lines are the values of Φ for the correspondence infiltration with root-water uptake. It can be seen that the values of Φ for infiltration without root-water uptake is higher than those with root-water uptake. This means that water content in the soil for infiltration without root-water uptake is higher than that in the soil for infiltration with root-water uptake. This result is expected as some amount of water in the soil absorbed by the plant roots for infiltration with root-water uptake. Thus, there is a decrease in the water content in the soil. The values of Φ increases as z increases, and convergent to a certain value of Φ. This means that water contents at shallower level of soil are lower than those at deeper. This is expected due to the assumption that there are no water fluxes across the surface of soil outside the channels. It can also be seen that Φ at $x = 0.385$ is the highest. This result indicate that water content at the location near the channels are higher than those further at the same level of soil depth.

The values of Φ for infiltration from semicircular and trapezoidal channels are about the same. However, the values of Φ for infiltration from triangular channels are lower than those from the others. This indicate that water contents in the soil for infiltration from triangular channels are the lowest.
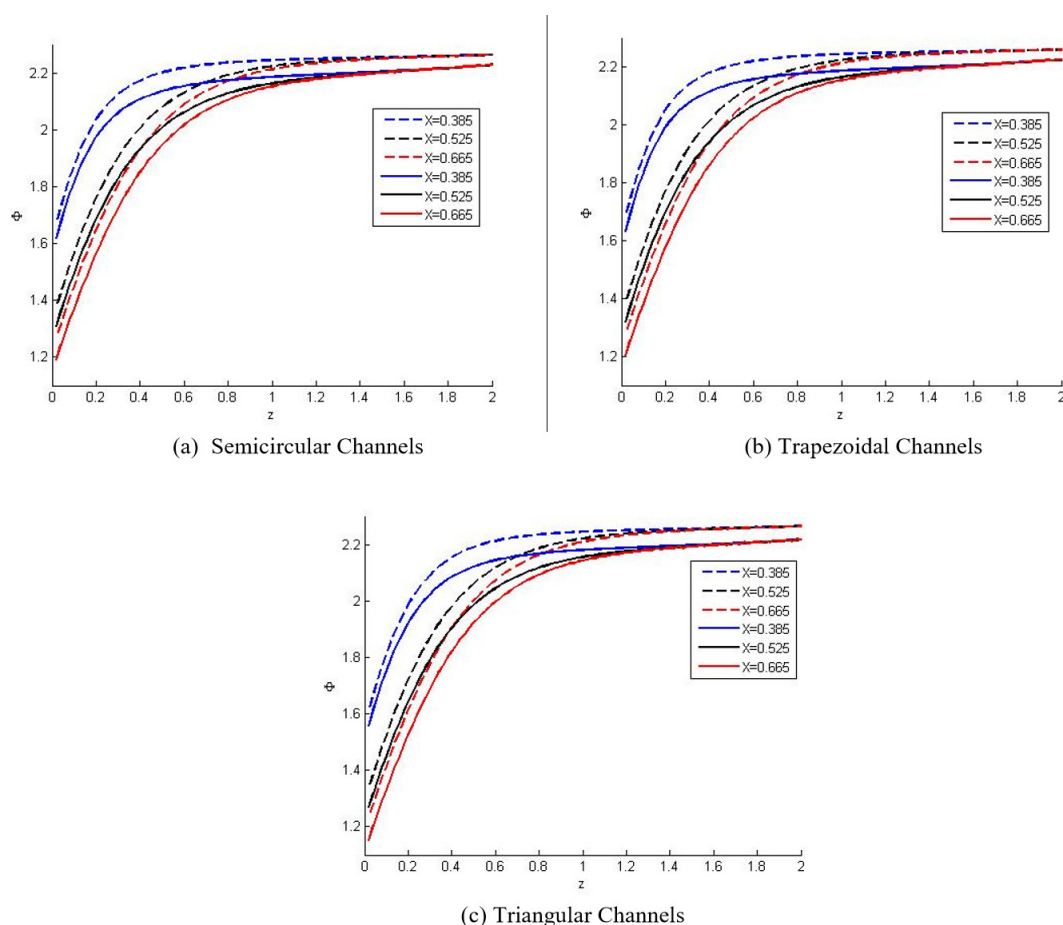
(a) Semicircular Channels

(b) Trapezoidal Channels

(c) Triangular Channels

Figure 3. Dimensionless MFP at selected values of x along z-axis.

Figure 4 shows values of suction potential, ψ, and corresponding values of root-water uptake, S, at three different values of X along the root zone. The three values of X are 55 cm, 75 cm, and 95 cm. The two values, $X = 55$ cm and $X = 95$ cm, are near the boundary of the root zone, and the other value, $X = 75$ cm, is in the middle of the root zone. It can be seen that the graph of ψ and S for infiltration from trapezoidal and semicircular channels have the same fashion. This means that water content in the soil as well as amount of water absorbed by plant roots for infiltration from trapezoidal and semicircular channels are about the same. For infiltration from triangular channels, the value of ψ are the smallest among those from the other types of channels, at any value of Z. Since the values of ψ are between -118 and -75, from Figure 3 higher ψ implies smaller value of response function, and hence smaller values of S as shown in Figure 5. This indicates that the amount of water absorbed by the roots from the soil with triangular channels is higher than those from the soil with other two channels. This result can clearly be observed from Figure 5. The highest uptake or water absorption for any values of X occurs at the surface of the soil. This is expected, as the values of X^* and Z^* are both 0 cm. This result indicates that the highest amount of water absorbed by the plant roots is at the distance of 0 cm from the plant.
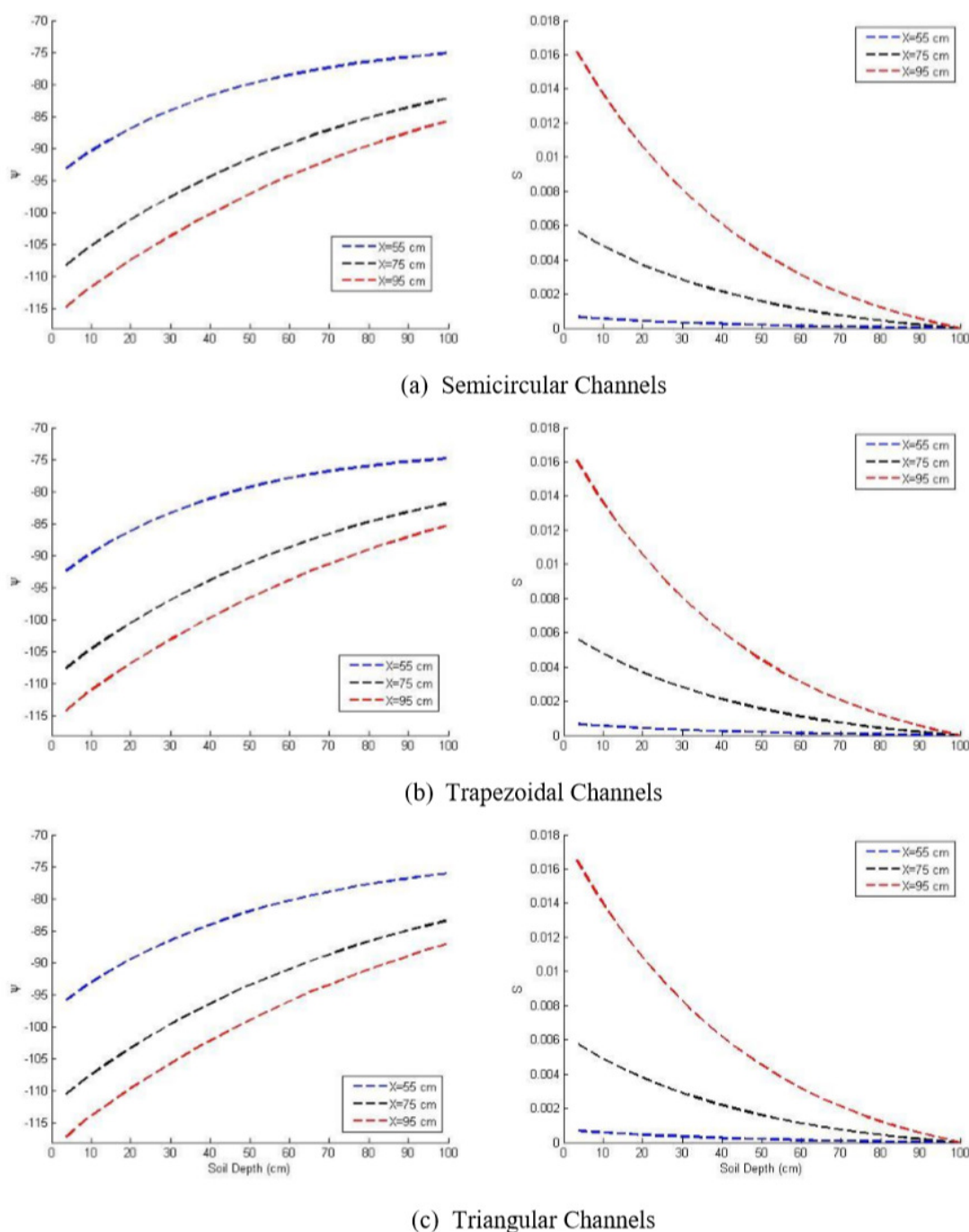
(a) Semicircular Channels

(b) Trapezoidal Channels

(c) Triangular Channels

Figure 4. Suction potential and corresponding root-water uptake at selected values of X.

5. Concluding Remarks

Problems involving steady infiltration from periodic channels with root-water uptake in a homogeneous soil have been solved numerically. Three different geometries of periodic channels with the same width and surface area were considered. The problems are solved numerically by employing a DRM with a predictor-corrector scheme. Using the method, numerical solutions of the dimensionless MFP are obtained. From the dimensionless MFP obtained and empirical parameters, suction potential can be calculated. Moreover, water uptake can also be calculated.

In this study, the results indicate that triangular channels provide lower suction potential than other two channels geometries. In contrast, triangular channels results in higher uptake of moisture than other two channels geometries.

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