The evolution of the bone in the half-plane under the influence of external pressure

O Gerasimov, F Shigapova, Yu Konoplev and O Sachenkov

Kazan Federal University, Institute of Mathematics and Mechanics, Russia, Tatarstan, Kazan, 18 Kremlevskaya St.

E-mail: 4works@bk.ru

Abstract. The paper deals with the problem of restructuring the trabecular bone tissue in the half-plane under the influence of a distributed load. We used evolutionary relations in Cowin's formulation, describing changes in pore orientation in terms of the fabric tensor and the fraction of the solid bone volume in the region under consideration. We modeled the process of the half-plane loading by the distributed pressure. The lazy zone theory and physical relations were used to obtain the initial value of the change in the solid bone volume fraction. As a result, we found the pore distribution field and represented it for clarity as a field of the first eigenvectors of the fabric tensor, which corresponds to the direction of an ellipse's larger radius. This study reveals variation of pore geometry in time and with the distance from the load action line as well as describes the behavior of the solid bone volume fraction with the distance from the pressure line.

1. Introduction

Description of porous structures is currently a topical problem in and solids mechanics [1-3] and bone biomechanics [4–6]. Recently, much attention has been paid to the types of pores in the bone tissue and the methods of describing their direction. Two types of pores are commonly distinguished: channel pores (vascular porosity) and pores formed by the rebuilding of trabeculae (lacunar-canalicular porosity). The use of the fabric tensor in description of such structures has lately become widespread [7–9]. It is known that the trabecular bone tissue is an inhomogeneous anisotropic porous structure, and its mechanical properties are largely determined by its internal structure. Many problems of the locomotive system require the stress-strain state of the bone tissue to be described taking into account the formation of its structure in time when the external load changes. For example, when setting problems of mechanical interaction in joints, it is necessary to consider peculiarities of the tissue under the articular surfaces, as well as to be able to evaluate structural changes during orthopedic interventions into the joint, since these changes can have a significant impact on the strength and stiffness properties of the bone [10, 11].

In this work, we consider the problem of evolution of the trabecular bone tissue and the problem of a half-plane loaded with distributed pressure. The initial state of the tissue is assumed to be uniformly filled by circular pores. The paper also examines the evolutionary model in Cowin's formulation [7–9].

2. Materials and Methods

Let us consider a half-plane (see figure 1) subjected to the distributed load q at the origin of coordinates. We will formulate the problem of the stress-strain state for the half-plane and the adaptive processes in it. Let us designate the region in question as S. Then the complete problem must contain the equilibrium equations:

$$\vec{\nabla}\tilde{\sigma}=0,\ \vec{x}\in S,\ t\geq 0 \tag{1}$$

physical relations [7–9]:

$$\tilde{\sigma} = (g_1 + g_2 e)tr\tilde{\varepsilon}\cdot E + (g_3 + g_4 e)\tilde{\varepsilon} + g_5(\tilde{\varepsilon}\tilde{K} + \tilde{K}\tilde{\varepsilon}) +$$
$$+ g_6\left(tr(\tilde{K}\tilde{\varepsilon})\cdot E + tr\tilde{\varepsilon}\cdot\tilde{K}\right), \vec{x}\in\overline{S}, t\geq 0, \tag{2}$$

where e is the variation of the solid bone volume, $\tilde{K}$ is the fabric tensor deviator, g_i are the elastic constants.

The evolutionary relation describing variation of the pore orientation in the region under consideration has the form:

$$\frac{d\tilde{K}}{dt} = h_1\left(\tilde{\varepsilon} - tr\tilde{\varepsilon}\cdot\frac{E}{3} - \left(\tilde{\varepsilon}_0 - tr\tilde{\varepsilon}_0\cdot\frac{E}{3}\right)\right) +$$
$$+ h_2\left(tr\left(\tilde{K}(\tilde{\varepsilon} - \tilde{\varepsilon}_0)\right)E - \frac{3}{2}\left(\tilde{K}(\tilde{\varepsilon} - \tilde{\varepsilon}_0) + (\tilde{\varepsilon} - \tilde{\varepsilon}_0)\tilde{K}\right)\right), \vec{x}\in\overline{S},\ t\geq 0. \tag{3}$$

The evolutionary relation describing variation in the fraction of the solid bone volume in the region under consideration has the form:

$$\frac{de}{dt}=(f_1 + f_2 e)(tr\tilde{\varepsilon} - tr\tilde{\varepsilon}_0),\ \vec{x}\in\overline{S},\ t\geq 0\quad. \tag{4}$$

Cauchy's geometrical relations:

$$\tilde{\varepsilon}=\frac{1}{2}\left(\vec{\nabla}\vec{u}+\vec{u}\vec{\nabla}\right),\ \vec{x}\in\overline{S},\ t\geq 0 \tag{5}$$

Boundary conditions:

$$\vec{n}\cdot\tilde{\sigma}=\vec{q},\ x\in S_q, t\geq 0$$
$$\vec{n}\cdot\tilde{\sigma}=0,\ x\notin S_q, t\geq 0 \tag{6}$$

Initial conditions:

$$\tilde{K}=\tilde{K}_0, e=e_0,\ x\in\overline{S}, t=0 \tag{7}$$

The stress field for the case of loading the half-plane is known and can be represented by the form:

$$\sigma_x\left(x,y\right)=\frac{q}{2\pi}\left(2\left(\theta_2 - \theta_1\right)+Sin2\theta_2 - Sin2\theta_1\right) \tag{8.1}$$

$$\sigma_y\left(x,y\right)=\frac{q}{2\pi}\left(2\left(\theta_2 - \theta_1\right)-Sin2\theta_2 + Sin2\theta_1\right) \tag{8.2}$$

$$\tau_{xy}(x,y) = \frac{q}{\pi}\left(Sin^2\theta_2 - Sin^2\theta_1\right)$$

(8.3)

where $\theta_1(x,y) = Arctg\left(\dfrac{q_s - x}{y}\right)$, $\theta_2(x,y) = Arctg\left(\dfrac{q_e - x}{y}\right)$.
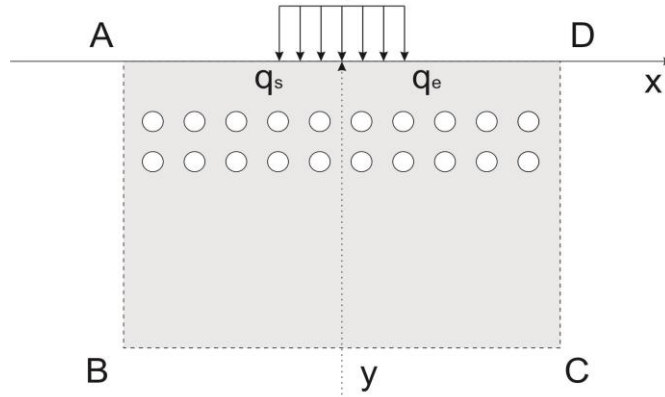

Figure 1. Computational scheme.

During the simulation, we assumed that at the initial time moment the bone is composed of uniform circular pores, and, therefore, the fabric tensor deviator is $\tilde{K}_0 = 0$. The following formal approach was used to determine the initial variation in the solid volume fraction. It is known that the reconstruction of the bone tissue does not occur when the strains are equal to $\tilde{\varepsilon}_{lz}$ ($\varepsilon_{ij} = -15 \cdot 10^{-4} \delta_{ij}$), i.e. in the case of the so-called lazy zone. Then, substituting the strain tensor of the lazy zone and the stress tensor in Equation (2), it is possible to determine the initial value e_0 of the solid bone volume fraction.

Since Equations (2)–(4) represent a system of nonlinear algebraic differential equations, the solution of the problem was implemented numerically. In this case, we considered the region ABCD (see figure 1), which was broken up into a regular grid where the equations were then solved. Numerical simulation was performed using the following values for the constants [7-9, 12]: $g_1 = 154.9$ GPa, $g_2 = 1147$ GPa, $g_3 = 612.9$ GPa, $g_4 = 4536$ GPa, $g_5 = 2384$ GPa, $g_6 = 510.8$ GPa, $h_1 = 0.01$ 1/day, $h_2 = 0.02$ 1/day, $f_1 = -2.5$ 1/day, $f_2 = 5$ 1/day. When making calculations, a single load was applied. Let us consider the properties of the fabric tensor, which is of interest in this work. The fabric tensor is by definition normalized in such a way that

$$Tr\tilde{H} = 1,$$

Its deviator can be defined by the formula

$$\tilde{K} = \tilde{H} - \frac{1}{3}\tilde{E}.$$

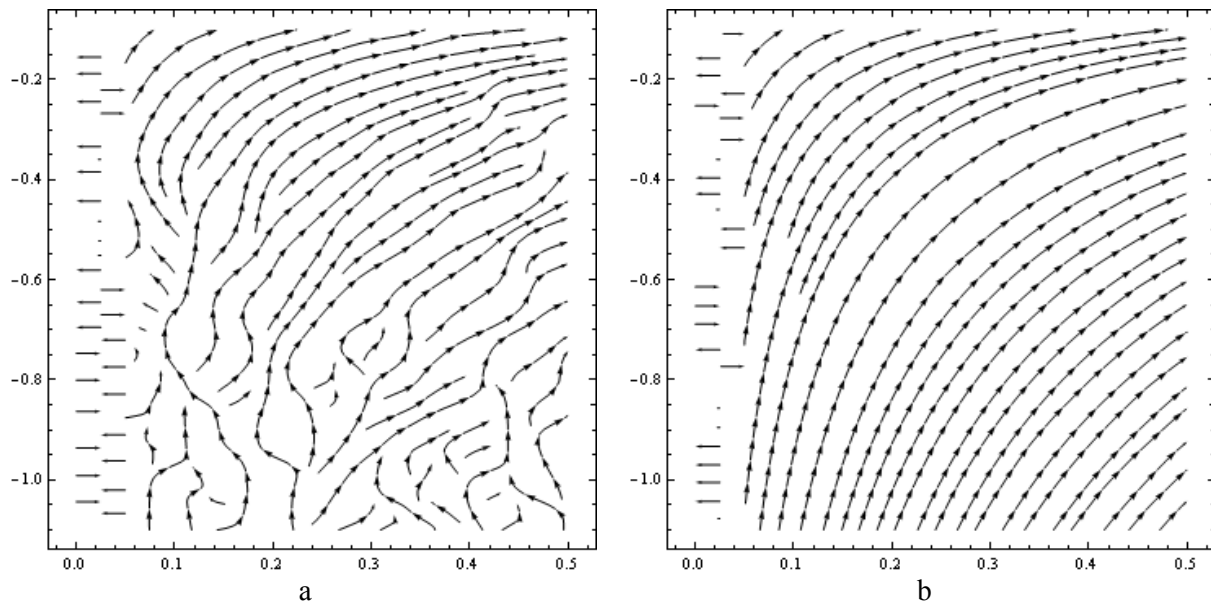As a result we obtain in this case

$$Tr\tilde{K} = 0.$$

Since the problem under consideration is a plane one, we assumed that the pores do not change their directions relative to the chart's plane; i.e. the fabric tensor deviator in its actual state can be represented as follows:

$$\tilde{K} = \begin{pmatrix} K_{11} & K_{12} & 0 \\ K_{12} & K_{22} & 0 \\ 0 & 0 & -K_{11} - K_{22} \end{pmatrix}.$$

3. Results

We made calculations for regions with a load which makes up 8–15% of their length (relation $q_s q_e/AD$ in figure 1). Considering the symmetry relative to the y axis, we show only the results for the positive abscissa axis. Let us present the results of calculation of adaptive processes for the fabric tensor. For clarity, we determined eigenvectors for the fabric tensor and built a field for the first eigenvector (see figure 2 a, b). The second eigenvector is directed orthogonally to the first. The geometrical meaning of these vectors is the direction of the semi-radii of elliptically shaped pores in the half-plane. To evaluate ellipticity, we built a distribution of the eigenvalues ratio for the fabric tensor (see figure 2 c, d).

At the beginning of the adaptation process, the pores begin to flatten under the load action line (see figure 2 a).When approaching the free surface, they begin to line up at an angle of about 30°, but at a distance they behave chaotically. The pores in this case still retain their original shape – the semi-radii ratio is 0.99 (see figure 2 c). With the increase of adaptation time, the direction of the pores changes: it remains the same under the load action line and is lined up along the hyperbolic lines at a distance (see figure 2 b). The pores get flattened by 15% in the region of load application (see figure 2 d); at a distance they remain close to round shape, the ratio of the semi0radii is equal to 0.93.
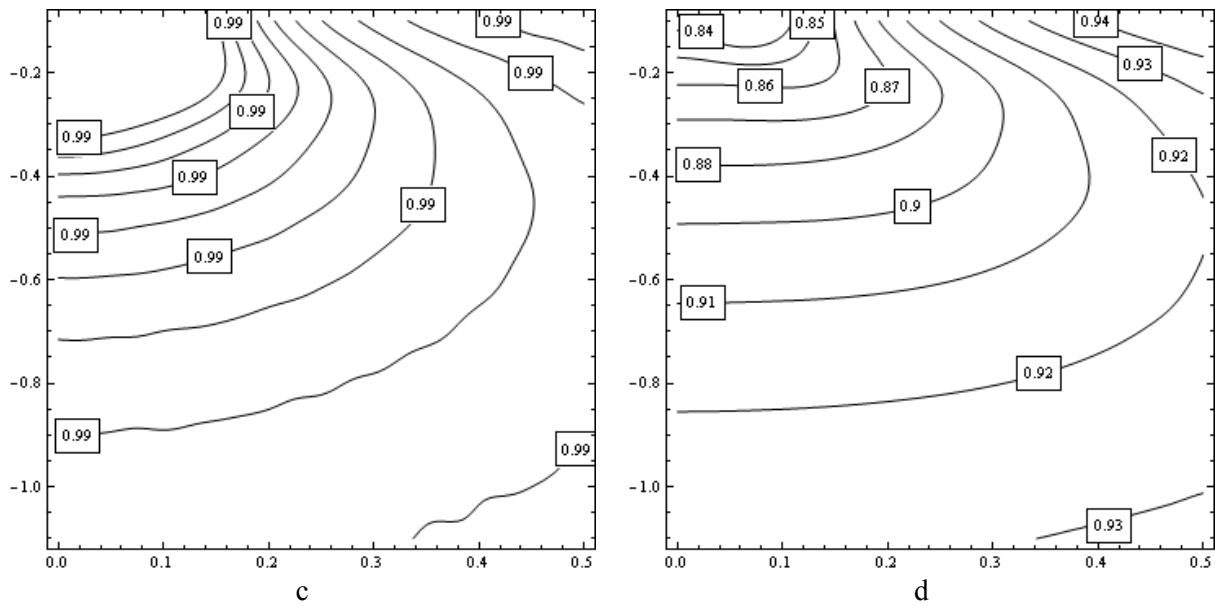

a

b

Figure 2. The fabric tensor calculation results: a – the directions of the first eigenvector at the beginning of the adaptation process; b – the directions of the first eigenvector at the end of the adaptation process; c – the ratio of the eigenvalues at the beginning of the adaptation process; d – the ratio of the eigenvalues at the end of the adaptation process.

Let us consider the results of the adaptation process. The variation in the solid bone volume fraction decreases dramatically down to certain minimum value, whereupon the solid volume fraction begins to increase slightly, reaching the asymptote. Figure 3a shows the distribution of changes in the fraction of the solid bone volume at the above-mentioned extreme moment. It is worth noting that all values are less than zero, which indicates that the porosity increases. If the process of adaptation continues, the values become asymptotic (see figure 3b), but remain negative. In this case the largest in magnitude values of variation in the solid bone volume fraction are at a distance from the applied load.
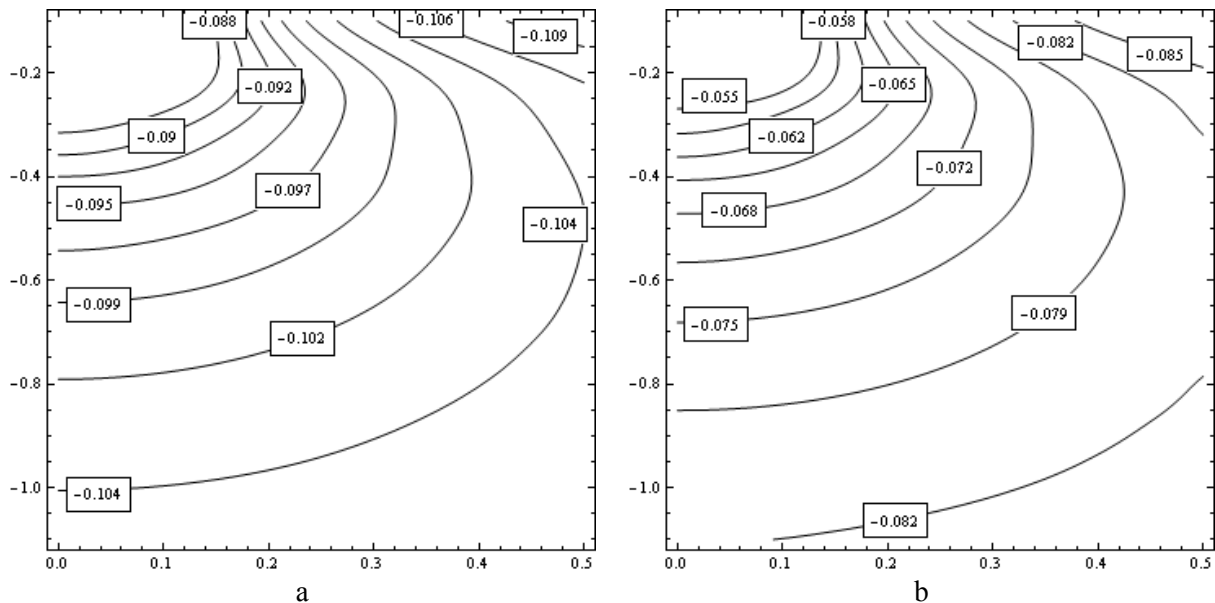

Figure 3. Calculation results for the variation in the solid bone volume fraction: a – the peak change in the solid bone volume fraction during the adaptation process, b – the variation of the solid bone volume fraction after the end of the adaptation process.

4. Discussion

As a result of simulation of adaptive processes for the case of a half-plane loaded by a distributed load, it was revealed that the pores become elongated under the load action line in the orthogonal direction; when getting remote from the load action line, the bigger semi-radii are directed along the hyperbolic curves, their asymptotes being the free surface and the load action line. The highest degree of ellipticity (semi-radii ratio) is achieved in the region of load application and decreases with the distance in the radial direction from zero. The variation of the solid bone volume fraction is negative, which shows the increase of porosity. We also found that the porosity increases sharply until a certain time, after which there is a smooth decrease of the porosity down to reaching the asymptote. The lowest value of porosity is observed in the under-load region and increases with the distance from zero in the radial direction.

5. Conclusion

This paper analyses the problem of restructuring the trabecular bone in a half-plane under the action of a distributed load, using evolutionary relations in Cowin's formulation, which describe the change in orientation of the pores in terms of the fabric tensor and the solid bone volume fraction in the region under consideration. We modeled the process of the half-plane loading by distributed pressure. Using the lazy zone theory and physical relations, we obtained the initial value of variation in the solid bone volume fraction. The field of pore distribution was determined as a result. For clarity it is represented as a field of the first eigenvectors of the fabric tensor, which corresponds to the direction of the bigger semi-radius of an ellipse. We reveal the nature of changes in the pore geometry with time and with the distance from the action line of the applied load and describe variation in the fraction of the solid bone volume with the distance from the pressure action line.

Acknowledgements

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