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Error estimations of mixed finite element methods for nonlinear problems of shallow shell theory

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Abstract. The variational formulations of problems of equilibrium of a shallow shell in the framework of the geometrically and physically nonlinear theory by boundary conditions of different main types, including non-classical, are considered. Necessary and sufficient conditions for their solvability are derived. Mixed finite element methods for the approximate solutions to these problems based on the use of second derivatives of the bending as auxiliary variables are proposed. Estimations of accuracy of approximate solutions are established.

1. Introduction

In present paper we propose and investigate a new class of finite element methods for solution of variational problems that describe the equilibrium of shallow shell with non-classical boundary conditions. These methods allow to use of simple Lagrangian finite elements and have an optimal error estimations.

2. Statement of the problem

The problem of equilibrium of a shallow shell in the framework of the geometrically and physically nonlinear theory can be formulated [1] as the problem of minimization of the potential energy functional

$$F(u) = \int\limits_{\Omega} \varphi(\varepsilon, \varkappa)dx - \int\limits_{\Omega} f \cdot u dx - \int\limits_{\Gamma} g \cdot u dx \tag{1}$$

on the space of displacements $u = (u_1, u_2, u_3)$ satisfying the main (geometric) boundary conditions. Here $u = (u_1, u_2, u_3)$ are tangent displacements and u_3 is the bending, Ω is a bounded domain on R^2 identified with the middle surface of the shell, referred to the Cartesian coordinate system, Γ is the boundary of Ω, f, g are given vector-functions describing the density of the external loads. Tangential and bending deformations of the shell are determined by the following relations: $\varepsilon = \varepsilon(u) = \{\varepsilon_{ij}(u)\}_{ij=1}^2$, $\varkappa = \varkappa(u) = \{\varkappa_{ij}(u)\}_{ij=1}^2$,

$$\varepsilon_{ij}(u) = e_{ij}(u) + k_{ij}u_3 + \frac{1}{2}\frac{\partial u_3}{\partial x_i}\frac{\partial u_3}{\partial x_j}, \quad e_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i, j = 1, 2, \tag{2}$$

$$\varkappa_{ij}(u) = -\frac{\partial^2 u_3}{\partial x_i \partial x_j}, \quad i, j = 1, 2, \tag{3}$$

where $k_{ij},\ i,j=1,2$ are initial curvatures of the shell. The real-valued function $\varphi(\xi)$, $\xi \in R^6$ is the density of the strain energy of the shell. It is assumed that the function φ is continuous and convex and there exist the constants c_0, c_1, $p \in (1,\infty)$ such that

$$c_0\,|\xi|^p \le \varphi(\xi) \le c_1\,|\xi|^p \quad \forall\,\xi \in R^6. \tag{4}$$

We introduce the Banach space $V = W_p^1(\Omega) \times W_p^1(\Omega) \times W_p^2(\Omega)$, where $W_p^l(\Omega)$, $l=1,2$, $p>1$, is the Sobolev space. We denote by V_0 the subspace of the space V obtained by closure of the space of the functions $C_0^\infty(\Omega) \times C_0^\infty(\Omega) \times C_0^\infty(\Omega)$. By V_τ we denote the subspace of the space V obtained by closure of the set of functions $C^\infty(\Omega) \times C^\infty(\Omega) \times C_0^\infty(\Omega)$ satisfying the boundary condition $\bar{u}(x)\cdot\nu(x)=0,\ x\in\Gamma$. Here ν is the unit normal field on Γ and $\bar{u}=(u_1,u_2)$ is the vector field of tangential displacements. Next, let V_ν be the subspace of the space V obtained by closure of the set of functions $C^\infty(\Omega) \times C^\infty(\Omega) \times C_0^\infty(\Omega)$ satisfying the boundary condition $\bar{u}(x)\cdot\tau(x)=0,\ x\in\Gamma$, where τ is the unit tangential field on Γ.

We suppose that $f \in [L_q(\Omega)]^3$, $g=(g_1,g_2)\in[L_q(\Gamma)]^2$, $1/p+1/q=1$, $k_{ij}\in L_{p_1}(\Omega)$, $p_1>p$.

The problem of minimization of the functional F on the space V_0 is called the Dirichlet problem (Problem I).

The problem of minimization of the functional F on the space V_τ it is natural to call the problem of hard contact of the border of the shell (Problem II).

The problem of minimization of the functional F on the space V_ν we call the problem of normal load on the boundary of the shell (Problem III).

3. Mixed finite element method

For each problem from I to III we propose mixed finite element methods of the type [2, 3] based on the use of second derivatives of functions as the auxiliary variables. Their approximate values $w_{ij}^h(y_3)\in H_l$ are defined by the following relations:

$$\int\limits_\Omega w_{ij}^h(y_3)\eta dx = -\frac{1}{2}\int\limits_\Omega \left(\frac{\partial y_3}{\partial x_i}\frac{\partial \eta}{\partial x_j} + \frac{\partial y_3}{\partial x_j}\frac{\partial \eta}{\partial x_i}\right) dx \quad \forall\,\eta \in H_l. \tag{5}$$

Here H_l is the finite element space, corresponding to polynomials of degree $l\ge 1$, and y_3 is the approximation of the bending. A characteristic feature of these schemes is that they are based on the simplest Lagrangian triangular or rectangular finite elements of the class C^0.

4. Investigation of mixed finite element method

The solvability of the corresponding discrete problems and convergence of their solutions can be establish under the same assumptions that are used in the justification of the solvability of the original variational problems [1].

Under additional assumptions on the function φ:

if $p\ge 2$ then

$$|\nabla\varphi(\xi)-\nabla\varphi(\eta)| \le c_1|\xi-\xi|(|\xi|+|\eta|)^{p-2} \quad \forall \xi,\eta\in R^6,$$

$$(\nabla\varphi(\xi)-\nabla\varphi(\eta))\cdot(\xi-\eta) \ge c_2|\xi-\eta|^p \quad \forall\xi,\eta\in R^6,$$

if $1<p<2$ then

$$|\nabla\varphi(\xi)-\nabla\varphi(\eta)| \le c_3|\xi-\xi|^{p-1} \quad \forall\xi,\eta\in R^6,$$

$$(\nabla\varphi(\xi)-\nabla\varphi(\eta))(|\xi|+|\eta|)^{2-p}\cdot(\xi-\eta) \ge c_4|\xi-\eta|^2 \quad \forall\xi,\eta\in R^6,$$

c_1,c_2,c_3,c_4 are positive constants, providing the implementation of inequalities of the type of strong convexity of the energy functional and the smoothness of the solution of the original problem we can establish the error estimates of the approximate solutions. The following estimates, which generalize the estimates of [4], are typical:

$$\sum_{ij=1}^{2}\left\|\frac{\partial^2 u_3}{\partial x_i \partial x_j} - w_{ij}^h(y_3)\right\|_{L_p(\Omega)} + \sum_{i=1}^{2}\|u_i - y_i\|_{W_p^1(\Omega)} \le ch^{(l-1)/(p-1)}, \quad p \ge 2,$$

$$\sum_{ij=1}^{2}\left\|\frac{\partial^2 u_3}{\partial x_i \partial x_j} - w_{ij}^h(y_3)\right\|_{L_p(\Omega)} + \sum_{i=1}^{2}\|u_i - y_i\|_{W_p^1(\Omega)} \le ch^{(l-2/p)/(p-1)}, \quad 1 < p < 2.$$

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