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On norms of operators generated by shift transformations arising in signal and image processing on meshes supplied with semigroup structures

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Abstract. Shift transformations and linear operators generated by shifts have a number of applications in signal and image processing. This note is concerned with a problem which has arisen in studying properties of real-world signals and images defined on meshes. For processing we suggest to introduce in domains of signals and images different semigroup structures. Semigroup operations give us opportunities to introduce shift transformations of signals and images. We study norms of polynomial filters generated by shift operators.

Introduction

We deal with a problem of operator theory which has arisen in studying properties of real-world signals and images. A part of motivation for our work has come from processing of experimental data in pressure pulse testing. Recall that a basic element in pulse testing is a well pair consisting of a pulsing well and an adjacent responding well [1]. There is a pair of signals, respectively. In practice, studying properties of these signals is often used for estimating a number of characteristics, for example, an average permeability. As is well known, different tools from functional analysis are involved in that investigation. For example, the Fourier transform is one such tool, and a wavelet expansion is another. The most basic linear operators in analysis of signals are shift transformations and their compositions. In particular, in time-frequency analysis these are the shifts in time and in frequency. Furthermore, there are a number of important operations in signal and image processing, such as the convolution and the correlation of signals, which use shift operators. It is also worth noting that shift transformations and linear operators generated by shifts play a significant role in different areas of functional analysis, in complex analysis and in dynamical systems. Another part of motivation for our work is a desire to give a self-contained elementary proof for a special case of the Coburn theorem from the theory of operator algebras (see, e.g., [2, Theorem 3.5.18]).

This note consists of Introduction and two Sections. In Section 1 we introduce the notations and recall some definitions and results that will be used in the sequel. In Section 2 we present results about norms of linear operators generated by shift transformations on the Hilbert space of square summable sequences.

1. Preliminaries

In practice signals and images are often defined on meshes of different types. We usually handle these meshes as sets without any additional structures.

Here, we suggest to consider a domain S of a signal $f : S \to \mathbb{F}$ as a mesh supplied with an additional structure, namely, a semigroup operation. In what follows, we shall make use of the additive notation. Here $\mathbb{F}$ denotes either the field of all real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$.

Let $l^2(S)$ be a Hilbert space of all square summable, in general, complex-valued signals:

$$l^2(S) = \{f : S \to \mathbb{F} : \sum_{s\in S} |f(s)|^2 < +\infty\}.$$

Recall that $l^2(S)$ is a Hilbert space under the coordinatewise linear operations and with the inner product defined by

$$< f, g >= \sum_{s\in S} f(s)\overline{g(s)}.$$

Let $\{e_s : s \in S\}$ be the canonical orthonormal basis in $l^2(S)$ given by the formula

$$e_a(b) = \begin{cases} 1, & \text{if } a = b; \\ 0, & \text{if } a \neq b. \end{cases}$$

A semigroup structure on a mesh allows us to introduce in consideration for every element $a \in S$ the shift operator defined by

$$T_a : l^2(S) \longrightarrow l^2(S) : e_b \longmapsto e_{a+b},$$

where $b \in S$.

In what follows, for simplicity, we assume that S is the semigroup of all non-negative integers $\mathbb{Z}^+ := \{0, 1, 2, \ldots\}$ or any its subsemigroup. In practice, for signal and image processing the author has also made use of the Cartesian products $\mathbb{Z}^+ \times \mathbb{Z}^+$ and $\mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{Z}^+$ with the natural operations and with some others in accordance with problems. Of course, those semigroups are used for one-dimensional, two-dimensional and three-dimensional signals, respectively. Moreover, the semigroups $\mathbb{Z}^+ \times \mathbb{Z}^+$ and $\mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{Z}^+$ and their subsemigroups have been used for matricisations and tensorisations of signals and images (see [3, Part II]).

In the case of the semigroup $S = \mathbb{Z}^+$, we have the right shift operator

$$T := T_1 : l^2(\mathbb{Z}^+) \longrightarrow l^2(\mathbb{Z}^+) : e_n \longmapsto e_{n+1}.$$

It is clear that the following infinite Toeplitz matrix M corresponds to the operator T with respect to the canonical basis $\{e_n : n \in \mathbb{Z}^+\}$:

$$M = \begin{pmatrix} 0 & 0 & 0 & 0\ldots \\ 1 & 0 & 0 & 0\ldots \\ 0 & 1 & 0 & 0\ldots \\ 0 & 0 & 1 & 0\ldots \\ 0 & 0 & 0 & 1\ldots \\ \vdots & \vdots & \vdots & \vdots\ddots \end{pmatrix}.$$

Of course, the right shift operator T is a linear bounded operator. Thus it is an element of the C^*-algebra $\mathcal{B}(l^2(\mathbb{Z}^+))$ of all bounded linear operators on the Hilbert space $l^2(\mathbb{Z}^+)$. Moreover, the operator T is a non-unitary isometry, i.e.,

$$\|Tx\| = \|x\|$$

for each $x \in l^2(\mathbb{Z}^+)$. Note that T is injective but not surjective.

It is easy to see that the adjoint T^* of T is the left shift operator defined by

$$T^*(e_n) = \begin{cases} e_{n-1}, & \text{if } n \geqslant 1; \\ 0, & \text{if } n = 0. \end{cases}$$

Obviously, we have $(T^*)^m = (T^m)^*$, $m \in \mathbb{Z}^+$. We denote this operator by T^{*m}.

One can easily see that the matrix of the left shift operator T^* with respect to the canonical basis is an infinite Toeplitz matrix which has the following form:

$$M^* = \begin{pmatrix} 0 & 1 & 0 & 0 \dots \\ 0 & 0 & 1 & 0 \dots \\ 0 & 0 & 0 & 1 \dots \\ 0 & 0 & 0 & 0 \dots \\ \vdots & \vdots & \vdots & \vdots \ddots \end{pmatrix}.$$

Note that the left shift operator T^* is surjective but not injective.

Let $I : l^2(\mathbb{Z}^+) \longrightarrow l^2(\mathbb{Z}^+)$ be the identity operator. Denote by $< e_0 >$ the one-dimensional linear subspace $\{\lambda e_0 : \lambda \in \mathbb{C}\}$ in $l^2(\mathbb{Z}^+)$. Let $Q_0 : l^2(\mathbb{Z}^+) \longrightarrow < e_0 >$ be the orthogonal projector onto $< e_0 >$:

$$Q_0(\sum_{i=0}^{\infty} \lambda_i e_i) = \lambda_0, \quad \text{where} \quad \sum_{i=0}^{\infty} \lambda_i e_i \in l^2(\mathbb{Z}^+).$$

Then we have the following relations for the composition operators:

$$T^*T = I, \tag{1}$$

$$TT^* = I - Q_0, \tag{2}$$

In other words, the composition TT^* is the orthogonal projector onto the subspace of $l^2(\mathbb{Z}^+)$ spanned by the vectors e_n, where $n = 1, 2, \dots$.

We consider the shifts T and T^* as operator generators of filters for signal and image processing on meshes with semigroup structures. Notice that some standard operations in digital signal processing have suitable analogs among those filters. That is, we have representations of operations as operators written by means of the shifts.

In practice, we have used these filters for denoising signals and in combination with matricisation and tensorisation operations of the Fourier transformation and a wavelet expansion for further signal and image processing.

2. Main results

We consider an arbitrary linear operator $p(T, T^*) \in \mathcal{B}(l^2(\mathbb{Z}^+))$ which is a polynomial in the variables T and T^* with complex coefficients. Equivalently, one can say that we take an element of the complex $*$-algebra $\mathcal{P}$ generated by T and T^*. It is worth noting that the algebra $\mathcal{P}$ is not complete. We call elements of $\mathcal{P}$ *polynomial filters*. We have the following lemma about representations of polynomial filters.

Lemma. *Every polynomial filter $p(T, T^*) \in \mathcal{P}$ can be represented as follows:*

$$p(T, T^*) = \lambda_1 T^{n_1} T^{*m_1} + \lambda_2 T^{n_2} T^{*m_2} + \dots + \lambda_d T^{n_d} T^{*m_d}, \tag{3}$$

where $\lambda_i \in \mathbb{C}$, (n_i, m_i) are distinct pairs of non-negative integers, $i = 1, \ldots, d$, and $d \in \mathbb{Z}^+ \setminus \{0\}$. Furthermore, for a non-zero polynomial filter representation (3) with non-trivial coefficients is unique up to an order of summands.

The sketch of the proof. The possibility to represent a polynomial filter as linear combination (3) follows from relations (1) and (2). To prove the uniqueness of representation (3) one can show that the family of operators $\{T^n T^{*m} : (n, m) \in \mathbb{Z}^+ \times \mathbb{Z}^+\}$ is linearly independent.□

Let us fix a number $k \in \mathbb{Z}^+ \setminus \{0\}$. For $j = 0, 1, \ldots, k-1$ we define the Hilbert spaces

$$H_j = \big\{x \in l^2(\mathbb{Z}^+) : x = \sum_{l=0}^{+\infty} \lambda_{j+kl} e_{j+kl}\big\},$$

which are subspaces of the Hilbert space $l^2(\mathbb{Z}^+)$. We also define the shift operators as follows:

$$T_j : H_j \longrightarrow H_j : e_{j+kl} \mapsto e_{j+k(l+1)},$$

where $l = 0, 1, 2, \ldots$

Below we present the results about polynomial filters and their norms.

Proposition 1. *Let $p(T, T^*)$ be a polynomial filter and $k \in \mathbb{Z}^+ \setminus \{0\}$. Then there exists a unitary operator $U : l^2(\mathbb{Z}^+) \longrightarrow \bigoplus_{j=0}^{k-1} H_j$ such that the following diagram*

$$\begin{tikzcd} l^2(\mathbb{Z}_+) \arrow[r, "p(T^k, T^{*k})"] \arrow[d, "U"'] & l^2(\mathbb{Z}_+) \arrow[d, "U"] \\ \bigoplus_{j=0}^{k-1} H_j \arrow[r, "\bigoplus_{j=0}^{k-1} p(T_j, T_j^*)"] & \bigoplus_{j=0}^{k-1} H_j, \end{tikzcd}$$

is commutative,i.e., we have the operator equality

$$p(T^k, T^{*k}) \circ U = U \circ \bigoplus_{j=0}^{k-1} p(T_j, T_j^*). \tag{4}$$

Standard properties of operators and their norms together with equality (4) imply

Corollary 1. *Let $p(T, T^*)$ be a polynomial filter and $k \in \mathbb{Z}^+ \setminus \{0\}$. Then the following equality holds:*

$$\|p(T^k, T^{*k})\| = \|p(T_0, T_0^*)\|.$$

In the similar way we have the next two statements.

Proposition 2. *Let $p(T, T^*)$ be a polynomial filter and $k \in \mathbb{Z}^+ \setminus \{0\}$. Then there exists a unitary operator $W : H_j \longrightarrow l^2(\mathbb{Z}^+)$ such that the following diagram*

$$\begin{tikzcd} H_0 \arrow[r, "p(T_0, T_0^*)"] \arrow[d, "W"'] & H_0 \arrow[d, "W"] \\ l^2(\mathbb{Z}_+) \arrow[r, "p(T, T^*)"] & l^2(\mathbb{Z}_+), \end{tikzcd}$$

is commutative,i.e., we have the operator equality

$$p(T, T^*) \circ W = W \circ p(T_0, T_0^*).$$

Corollary 2. *Let $p(T, T^*)$ be a polynomial filter and $k \in \mathbb{Z}^+ \setminus \{0\}$. Then the following equality holds:*

$$\|p(T, T^*)\| = \|p(T_0, T_0^*)\|.$$

As an immediate consequence of Corollaries 1 and 2 we obtain

Corollary 3. *Let $p(T, T^*)$ be a polynomial filter and $k \in \mathbb{Z}^+ \setminus \{0\}$. Then the following equality holds:*

$$\|p(T^k, T^{*k})\| = \|p(T, T^*)\|.$$

Let $\mathcal{T}$ be the C^*-subalgebra of $\mathcal{B}(l^2(\mathbb{Z}^+))$ generated by the operator T. The C^*-algebra $\mathcal{T}$ is called *the Toeplitz algebra generated by the semigroup* , or *the reduced semigroup C^*-algebra of* $\mathbb{Z}^+$. Note that the $*$-algeba $\mathcal{P}$ is dense in the C^*-algebra $\mathcal{T}$ with respect to the norm topology of $\mathcal{B}(l^2(\mathbb{Z}^+))$. From the above statements the following special case of the Coburn theorem follows:

Proposition 3. *Let $k \in \mathbb{Z}^+ \setminus \{0\}$. Then there exists a unique $*$-endomorphism $\varphi : \mathcal{T} \longrightarrow \mathcal{T}$ of the Toeplitz algebra such that $\varphi(T) = T^k$.*

The sketch of the proof. The proof consists of two steps. First, we construct the $*$-homomorphism $\varphi_0 : \mathcal{P} \longrightarrow \mathcal{T}$ which is well-defined in view of Lemma by the formula

$$\varphi_0(T^n T^{*m}) = T^{kn} T^{*km}.$$

Second, using Corollary 3, we extend the $*$-homomorphism φ_0 by continuity to the $*$-endomorphism on the whole space $\mathcal{T}$. $\square$

References

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