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Non-overlapping domain decomposition method for a variational inequality with gradient constraints

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Abstract. Non-overlapping domain decomposition method is applied to a variational inequality with nonlinear diffusion-convection operator and gradient constraints. The method is based on the initial approximation of the problem and its subsequent splitting into subproblems. For the resulting constrained saddle point problem block relaxation-Uzawa iterative solution method is applied.

1. Introduction

Domain decomposition methods for the variational inequalities have been investigated for a long time. The most attention has been paid to Schwarz-type iterative methods for the problems with pointwise constraints to solution [1-12]. This type of problems includes obstacle problems, some contact problems, Stefan problem on a fixed time level among others. Non-overlapping domain decomposition method has been applied to variational inequalities with constraints to a solution in the supposition that the location of free boundary is known [13-15].

Convergence of Uzawa-type iterative methods for the constrained saddle point problems has been investigated in [16-18]. First general result on the convergence of Uzawa iterative method has been proved in the article [16], where sufficient convergence condition has been formulated in terms of matrices inequality. In [17] a generalization of this result for wider class of saddle point problems and for so-called block relaxation-Uzawa iterative solution method have been investigated. These results were applied to iterative solution methods for mesh variational inequalities with gradient constraints and for mesh approximations of state and control constrained optimal control problems in the numerous subsequent articles.

In this article we apply the aforementioned results on the iterative solution methods for the constrained saddle point problems to non-overlapping domain decomposition method for variational inequalities with gradient constraints.

2. Variational inequality and its approximation

Let $\Omega \subset R^2$ with be a bounded domain with a piecewise smooth boundary $\partial\Omega$, $V = H_0^1(\Omega)$ be Sobolev space with the norm $\|u\| = \left(\int_\Omega |\nabla u|^2 dx\right)^{1/2}$. Further, c_f is a constant in Friedrichs inequality $\|u\|_{L_2(\Omega)} \le c_f \|u\| \quad \forall\, u \in V.$

Define the functions

$$f \in L_2(\Omega), a, b \in L_\infty(\Omega), \quad where \quad a(x) \ge a_0 > 0 \quad a.e.\, in\, \Omega, \tag{1}$$

and $g_1(\bar{t}), g_2(s,\bar{t})$, which for all $s \in R, \bar{t} \in R^2$ satisfy the following assumptions:

$$\begin{cases} g_1(\bar{t}) \text{ and } g_2(s,\bar{t}) \text{ are continuous}, \left|g_1(\bar{t})\right| \le c\left|\bar{t}\right|, \\ |g_2(s_1,\bar{t}) - g_2(s_2,\bar{t})| \le \beta_1\,|s_1 - s_2| + \beta_2\,|\bar{t}_1 - \bar{t}_2|, \\ (g_1(\bar{t}_1) - g_1(\bar{t}_2), \bar{t}_1 - \bar{t}_2) \ge \sigma_0\,|\bar{t}_1 - \bar{t}_2|^2, \\ \sigma_0 > 0,\, a_0\sigma_0 - b\beta_1 c_f^2 - b\beta_2 c_f \equiv \sigma > 0,\, b = \sup_{x\in\Omega} |b(x)|. \end{cases} \tag{2}$$

Define a semilinear form $a(\cdot,\cdot): V \times V \to R$ by the equality

$$a(u,v) = \int_\Omega a(x) g_1(\nabla u)\cdot\nabla v\, dx + \int_\Omega b(x) g_2(u,\nabla u) v\, dx.$$

Due to assumptions (1), (2) the form $a(u,v)$ is continuous on $V \times V$ and uniformly monotone:

$$a(v, v-u) - a(u, v-u) \ge \sigma\|u - v\|^2.$$

Consider variational inequality

$$u \in V: \quad a(u, v-u) + \int_\Omega |\nabla v|\, dx - \int_\Omega |\nabla u|\, dx \ge \int_\Omega f(v-u)\, dx \quad \forall\, v \in V. \tag{3}$$

Proposition 1. *Under the assumptions (1), (2) variational inequality (3) has a unique solution.*

The proof is based on the theory of variational inequalities with monotone operators [19].

Let Ω be a polygonal domain. We approximate variational inequality (3) by using first order finite element method on a triangle grid (cf [20]-[22]). Let T_h be a conforming triangulation of $\overline{\Omega}$ into triangle finite elements e, $V_h = \{u_h \in H_0^1(\Omega) \cap C(\overline{\Omega}): u_h \in P_1 \quad on \quad every \quad e \in T_h\}$ be the space of the continuous and piecewise linear functions, while $U_h = \{u_h \in L_2(\Omega): u_h \in P_0 \quad on \quad every \quad e \in T_h\}$ be the space of the piecewise constant functions. Consider f_h, a_h and b_h are functions from U_h which equal to mean values of f, a and b on every $e \in T_h$. Namely, $f_h = |e|^{-1} \int_{t\in e} f(t)\, dt$ for every $e \in T_h$, $|e| = mes\, e$, and similarly for a_h, b_h. Let us define

$$a_h(u_h, v_h) = \int_\Omega a_h g_1(\nabla u_h)\cdot\nabla v_h\, dx + \int_\Omega b_h g_2(u_h, \nabla u_h) v_h\, dx.$$

Due to assumptions (1), (2) the form $a_h(u_h, v_h)$ is continuous on $V_h \times V_h$ and uniformly monotone. Discrete variational inequality, approximating (3):

$$u_h \in V_h: \quad a_h(u_h, v_h - u_h) + \int_\Omega |\nabla v_h|\, dx - \int_\Omega |\nabla u_h|\, dx \ge \int_\Omega f_h(v_h - u_h)\, dx \quad \forall\, v_h \in V_h. \tag{4}$$

Proposition 2. *Under the assumptions (1), (2) variational inequality (4) has a unique solution.*

3. Domain decomposition and constructing a saddle point problem

Let us decompose the domain Ω into m subdomains $\Omega = \sum_{i=1}^{m} \Omega_i + \bigcup_{i \neq j} \Gamma_{ij}$, where every open subdomain Ω_i consists of the elements $e \in T_h$ and $\Gamma_{ij} \in \Omega$ is the general part of the boundaries of $\partial\Omega_i$ and $\partial\Omega_j$ lying inside Ω. Hereafter we use the symbol Σ for the non-intersected sets (above $\Omega_i \bigcap \Omega_j = \varnothing$ for all $i \neq j$).

By $\Gamma = \bigcup_{i \neq j} \Gamma_{ij}$ we denote the skeleton of the decomposition. The boundary of a subdomain generally consists of two parts: $\partial\Omega_i = \Gamma_{i0} + \Gamma_i$ with $\Gamma_{i0} \subset \partial\Omega,\ \ \Gamma_i \subset \Gamma$. The case $\Gamma_{i0} = \varnothing$ is allowed.

We use the following spaces of mesh functions: $V_{hi} = \{u_{hi}$ *is the restriction of* $u_h \in V_h$ *to* $\overline{\Omega}_i\}$, $\overline{V}_h = \{\overline{u}_h = (u_{h1}, u_{h2}, \ldots, u_{hm}),\ \ u_{hi} \in V_{hi}\}$. W_h is the space of traces of functions from V_h to skeleton Γ. $K_h = \{(\overline{u}_h, s_h) \in \overline{V}_h \times W_h:\ \ u_{hi} = s_h \ \ on\ \ \Gamma_i,\ \ i = 1,2,\ldots,m\}$.

Let

$$a_{hi}(u_h, v_h) = \int_{\Omega} a_h g_1(\nabla u_h) \cdot \nabla v_h \, dx + \int_{\Omega} b_h g_2(u_h, \nabla u_h) v_h \, dx,$$

$$\overline{a}_h(\overline{u}_h, \overline{v}_h) = \sum_{i=1}^{m} a_{hi}(u_{hi}, v_{hi}) \quad for \quad \overline{u}_h \in \overline{V}_h.$$

Similarly define $\varphi_{hi}(u_{hi}),\ f_{hi}$ on the subdomains and composite functions $\overline{\varphi}_h(\overline{u}_h),\ \overline{f}_h$.

Consider variational inequality: find $(\overline{u}_h, s_h) \in K_h$ such that

$$\overline{a}_h(\overline{u}_h, \overline{v}_h - \overline{u}_h) + \overline{\varphi}_h(\overline{v}_h) - \overline{\varphi}_h(\overline{u}_h) \geq \overline{f}_h(\overline{v}_h - \overline{u}_h) \quad \forall\, (\overline{v}_h, s_h) \in K_h. \tag{5}$$

Proposition 3. *Problem (5) has a unique solution* $(\overline{u}_h, s_h) \in K_h$ *such that* u_{hi} *is the restriction to* Ω_i *of the solution* u_h *of problem (4), while* s_h *is the trace of* u_h *on the skeleton* Γ.

We put in the correspondence to a mesh function the vector of its nodal values. In particular, let $u_i \in R^{N_i}$ be the vector of nodal values of a function $u_{hi} \in V_{hi}$, and we use notation $u_i \leftrightarrow u_{hi}$ for this correspondence. Vector $u = (u_1, u_2, \ldots, u_m)^T$ with $u_i \in R^{N_i}$ corresponds to $\overline{u}_h \in \overline{V}_h$ ($\overline{u} \leftrightarrow \overline{u}_h$), while $\overline{q}$ corresponds to $\overline{q}_h = (q_{1h}, q_{2h}) \in U_h \times U_h$.

Below by $(\cdot,\cdot)_i$ we mean the Euclidian scalar product in R^{n_i} and denote $(u, v) = \sum_{i=1}^{m} (u_i, v_i)$ for $u = (u_1, u_2, \ldots, u_m)^T$ and $v = (v_1, v_2, \ldots, v_m)^T$.

Let for every i=1,2,...,m the following matrices and nonlinear operators be defined by the forms on the subdomains:

$$(\tilde{L}_i u_i, q_i)_i = \int_{\Omega_i} \nabla u_{hi}(x) \cdot \overline{q}_{hi}(x)\, dx, \quad (M_{pi} p_i, q_i)_i = \int_{\Omega_i} \overline{p}_{hi}(x) \cdot \overline{q}_{hi}(x)\, dx,$$

$$(M_{ui} u_i, v_i)_i = \int_{\Omega_i} u_{hi}(x) \cdot v_{hi}(x)\, dx, \quad (k_{1i}(p_i), q_i)_i = \int_{\Omega_i} a_h(x)\, g_1(\overline{p}_h(x)) \overline{q}_h(x)\, dx,$$

$$(k_{2i}(u_i, p_i), v_i)_i = \int_{\Omega_i} b_h(x)\, g_2(u_h, \overline{p}_h(x)) v_h(x)\, dx, \quad L_i = M_{pi}^{-1} \tilde{L}_i.$$

The corresponding composite matrices and operators

$$Lu = \sum_{i=1}^{m} L_i u_i, \quad L^T k_1(Lu) = \sum_{i=1}^{m} L_i^T k_{1i}(L_i u_i), \quad k_2(u, Lu) = \sum_{i=1}^{m} k_{2i}(u_i, L_i u_i)$$

have block diagonal forms.

Using the introduced notations we get the equality

$$\bar{a}_h(\bar{u}_h, \bar{v}_h) = (k_1(Lu), Lv) + (k_2(u, Lu), v) = (N(u), v) \quad \text{for} \quad \bar{u}_h \leftrightarrow u, \bar{v}_h \leftrightarrow v$$

with block diagonal, continuous and monotone operator $N(u) = L^T k_1(Lu) + k_2(u, Lu)$. Denote by $\theta_i(L_i u_i) = \int_{\Omega_i} |\nabla u_{hi}|\, dx \quad \text{for} \quad u_i \leftrightarrow u_{hi}$, and $\theta(Lu) = \sum_{i=1}^{m} \theta_i(L_i u_i)$.

Finally, $(\bar{u}_h, s_h) \in K_h \leftrightarrow (u, s) \in K = \{R_i u_i = S_i s, \quad i = 1,2,...,m\}$, where $R_i u_i$ is the trace of u_i on Γ_i, while $S_i s$ is the restriction of *s* to Γ_i.

Now, variational inequality (5) can be written as the following variational inequality for the vectors of nodal values of corresponding mesh functions:

$$(u, s) \in K: \quad (N(u), v - u) + \theta(Lv) - \theta(Lu) \geq f(v - u) \quad \forall\, (v, t) \in K. \tag{6}$$

Obviously, this variational inequality has a unique solution because (5) has a unique solution.

Proposition 4. *A pair $(u, s) \in K$ is a solution of (6) if and only if there exists a vector μ such that the triple (u, s, μ) satisfies the following saddle point problem*:

$$\begin{pmatrix} N + L^T \circ \partial\theta \circ L & 0 & -R^T \\ 0 & 0 & S^T \\ -R & S & 0 \end{pmatrix} \begin{pmatrix} u \\ s \\ \mu \end{pmatrix} \ni \begin{pmatrix} f \\ 0 \\ 0 \end{pmatrix}, \tag{7}$$

where $R = diag(R_1, R_2, \ldots R_m)$, $S = (R_1, R_2, \ldots R_m)^T$ and $\partial\theta$ is the subdifferential of the convex function.

To apply Uzawa-type iterative method we make several equivalent transformations of saddle point problem (7). First, define the auxiliary vectors $p=Lu$ and $\lambda \in k_1(p) + \partial\theta(p)$. Then the inclusion $N(u) + L^T \partial\theta(Lu) - f \ni R^T \mu$ in the first row of (7) transforms to the system with respect to the vectors (u, p, λ): $k_2(u, p) + L^T \lambda - R^T \mu = f, \quad k_1(p) + \partial\theta(p) \ni \lambda, \quad Lu = p$.

Problem (7) turns to the following one: find $w = (u, p, s)^T, \quad \eta = (\lambda, \mu)^T$ such that

$$\begin{pmatrix} A_0 & B \\ -B^T & 0 \end{pmatrix} \begin{pmatrix} w \\ \eta \end{pmatrix} \ni \begin{pmatrix} F \\ 0 \end{pmatrix}. \tag{8}$$

with right-hand side $F=(f,0,0)^T$, matrix $B=\begin{pmatrix} L & -E & 0\\ -R & 0 & S\end{pmatrix}$, E is identity matrix, and non-linear operator $A_0(w)=(k_2(u,p),k_1(p)+\partial\theta(p),0)^T$ for $w=(u,p,s)^T$.

Since the operator A_0 is degenerate, we make further equivalent transformation of system (8) by using the equations *Lu-p*=0 and *Ru-Ss*=0. These transformations result in the following saddle point problem

$$\begin{pmatrix} A & B\\ -B^T & 0\end{pmatrix}\begin{pmatrix} w\\ \eta\end{pmatrix}\ni\begin{pmatrix} F\\ 0\end{pmatrix} \tag{9}$$

with $A(w)=((rL^TM_pL+r_1R^TR)u-rL^TM_pp+k_2(u,p)-r_1R^TSs,k_1(p)+\partial\theta(p),-r_1S^TRu+r_1S^TSs)^T$ and positive parameters r and r_1 which are defined to ensure the uniform monotonicity of A.

Proposition 5. *If $r_1>0$ while parameter r satisfies the inequalities*

$$0<2a_0\sigma_0-b\beta_2c_f-2\sqrt{a_0\sigma_0\sigma}<r<2a_0\sigma_0-b\beta_2c_f+2\sqrt{a_0\sigma_0\sigma}\,, \tag{10}$$

then operator A is uniformly monotone:

$$\exists\,\gamma_1>0:\quad (A(w_1)-A(w_2),w_1-w_2)_w\ge\gamma_1\|w_1-w_2\|_w^2. \tag{11}$$

Proposition 6. *Problem (9) has a solution (u,p,s,λ,μ) with unique components (u,p,s).*

4. Iterative solution method

Now let us fix a parameter $r_1>0$ and $r_0=2\sigma_0-\beta_2c_f$ (the middle point of the admissible interval). Define a non-linear operator $A(w,\tilde{w})$ for the vectors $w=(u,p,s)^T$ and $\tilde{w}=(\tilde{u},\tilde{p},\tilde{s})^T$ by the equality

$$A(w,\tilde{w})=((rL^TM_pL+r_1R^TR)u-rL^TM_pp+k_2(\tilde{u},p)-r_1R^TS\tilde{s},k_1(p)+\partial\theta(p),-r_1S^TRu+r_1S^TSs)^T.$$

Proposition 7. *If $D_1=\begin{pmatrix} M_p^{-1} & 0\\ 0 & E\end{pmatrix}$ and $\tau<r_0$, then there exists $\alpha>1,\quad\gamma>0$ such that*

$$(A(w_1,\tilde{w}_1)-A(w_2,\tilde{w}_2),w_1-w_2)_w-\frac{\alpha\tau}{2}(D^{-1}B(w_1-w_2),B(w_1-w_2))\ge \tag{12}$$

$$\ge\gamma\|w_1-w_2\|_w^2+\rho(w_1-w_2)-p(\tilde{w}_1-\tilde{w}_2),\quad \rho(w)=0.5b\beta_1c_f^2(M_pLu,Lu)+0.5r_1\|Ss\|^2.$$

Block relaxation-Uzawa method with a preconditioner D for solving (9) reads as follows:

$$A(w^{k+1},w^k)\ni F,$$

$$D\frac{\eta^{k+1}-\eta^k}{\tau}+Bw^{k+1}=0. \tag{13}$$

Due to the inequalities (11) and (12) and general convergence Theorem 1 from [16] we get the following result:

Theorem 1. If $D=\begin{pmatrix} M_p^{-1} & 0\\ 0 & E\end{pmatrix}$ and $\tau<r_0$ then iterative method (13) converges from any initial guess w^0.

The algorithm to implement method (13) reads as follows: given an initial guess (u^0,λ^0,s^0) for $k=0,1...$ calculate sequently

$$k_1(p^{k+1}) + \partial\theta(p^{k+1}) \ni \lambda^k,$$
$$(rL^T M_p L + r_1 R^T R)u^{k+1} = rL^T M_p p^{k+1} - k_2(u^k, p^{k+1}) - r_1 R^T S s^k + f - L^T \lambda^k + S^T \mu^k,$$
$$r_1 S^T S s^{k+1} = r_1 S^T R u^{k+1} - S^T \mu^k,$$
$$\lambda^{k+1} = \lambda^k + \tau M_p(p^{k+1} - Lu^{k+1}),$$
$$\mu^{k+1} = \mu^k + \tau(Ru^{k+1} - Ss^{k+1}).$$

Thus, we have to solve on every iteration the system of inclusions to find p^{k+1} and then the systems of linear equations to find u^{k+1} and s^{k+1}. We emphasize that owing to the block diagonal form of the operators and matrices the inclusion for p^{k+1} and equation for u^{k+1} are split up into uncoupled systems, corresponding to the subdomains. Moreover, since every operator $k_{1i} + \partial\theta_i$ for $i = 1,2,\ldots,m$ has block diagonal form with 2×2 blocks, we can easily find the exact solutions of the corresponding inclusions in explicit forms.

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