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Method of investigation of deformations of solids of incompressible materials

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Abstract. The aim of this work is development mathematical models, algorithm for the investigation stress-strain state of elastic solids, taking into account the incompressibility materials. The constitutive equations are received using a potential energy of deformations. The system of the linear algebraic equations is received by linearization of a resolving equation. The penalty method is applied for a modelling of the incompressibility of the material. The finite element method is used for numerical solution of the problems.

1. Introduction

The problem of mathematical modelling of elastic incompressible materials, which are referred to rubber and the other elastomers, is of great significance in the modern solid mechanics. From the point of view of mechanics, such problems are the objects of researches of non-linear theory of elasticity. This is due to their property that they allow large deformations, while maintaining the elastic properties. The authors often use numerical methods to solve non-linear problems [1–18]. The objective of our work is development of mathematical model and numerical realization of algorithm of research of stress-strained state of elastic bodies in terms of incompressibility of material and large deformations.

2. Kinematics. Constitutive equations

Kinematics of finite deformation is described using [11–14, 19]: deformation gradient tensor $\left(\boldsymbol{F}\right)$ left Cauchy–Green tensor (Finger tensor) $\left(\boldsymbol{B}\right)$ spatial velocity gradient tensor $\left(\boldsymbol{h}\right)$; the deformation rate tensor $\left(\boldsymbol{d}\right)$.

A stored strain energy function or elastic potential W, which depends on the Finger tensor can be determined as

$$W = W\left(I_{1B}, I_{2B}, I_{3B}\right),$$

where I_{1B}, I_{2B}, I_{3B} – invariants of the tensor $\left(\boldsymbol{B}\right)$.

The stress state is characterized by a Cauchy–Euler stress tensor, which is defined in the current state [20–27]:

$$\left(\boldsymbol{\sigma}\right) = \sigma_{ij}\left(\boldsymbol{e}_i \boldsymbol{e}_j\right).$$

Constitutive relationships are written in the following form:

$$\left(\boldsymbol{\sigma}^{Tr}\right)=(\boldsymbol{c})\cdot\cdot(\boldsymbol{d}),$$

where $\left(\boldsymbol{\sigma}^{Tr}\right)=(\dot{\boldsymbol{\sigma}})+(\boldsymbol{h})\cdot(\boldsymbol{\sigma})+(\boldsymbol{\sigma})\cdot(\boldsymbol{h})^T-I_{1d}(\boldsymbol{\sigma}),(\dot{\boldsymbol{\sigma}})=(\boldsymbol{c})\cdot\cdot(\boldsymbol{d})+(\boldsymbol{h})\cdot(\boldsymbol{\sigma})+(\boldsymbol{\sigma})\cdot(\boldsymbol{h})^T-(\boldsymbol{\sigma})I_{1d}$,
$\mathbf{c}=\frac{4}{J}(\boldsymbol{B})\cdot\frac{\partial^2 W}{\partial\boldsymbol{B}\partial\boldsymbol{B}}\cdot(\boldsymbol{B})$ – elasticity tensor, $J=\frac{d\Omega}{d\Omega_0}$ – a changing of volume, Ω – current volume, Ω_0 – initial volume.

We applied the penalty method for solving of problems with incompressibility. The material is considered as nearly incompressible, and the pressure introduced into the equation as an independent variable:

$$P=k(J-1).$$

When the penalty number (bulk modulus) $k\to\infty$, $J\to 1$, which means that the condition of incompressible material $J=1$.

For an isotropic nearly compressible material, the modified deformation gradient and left Cauchy-Green tensor are considered:

$$\hat{\mathbf{F}}=J^{-1/3}\mathbf{F},\quad \hat{\mathbf{B}}=J^{-2/3}\mathbf{B}.$$

For these measures $J=I_{3\hat{\mathbf{B}}}=1$.

Thus, the strain energy density is represented in the form of a sum of two terms; the first depends on the changes in volume, and the second – on the invariants of the measures of deformations:

$$W=W\left(I_{1\hat{B}},I_{2\hat{B}}\right)+W(J)=W\left(I_{1\hat{B}},I_{2\hat{B}}\right)+\frac{k}{2}(J-1)^2.$$

3. Variational formulation

The algorithm of investigation is based on an Update Lagrange formulation. The equation of principle of virtual work in terms of the virtual velocity in the current configuration is used [27–29]:

$$\int_{\Omega}(\boldsymbol{\sigma})\cdot\cdot(\delta\boldsymbol{d})d\Omega=\int_{S^\sigma}\boldsymbol{t}_n\cdot\delta\boldsymbol{v}ds+\int_{\Omega}\boldsymbol{f}\cdot\delta\boldsymbol{v}d\Omega,$$

where $\vec{\boldsymbol{v}}$ – velocity vector of a material point; S^σ – part of the surface on which are defined forces; $\vec{\boldsymbol{t}}_n$, $\vec{\boldsymbol{f}}$ – vectors of surface and body forces respectively.

After linearization, we received the resolving system of the linear algebraic equations:

$$\int_{\Omega_k}\{\left({}^k\boldsymbol{d}\right)\cdot\cdot\left({}^k\boldsymbol{c}\right)\cdot\cdot(\delta\boldsymbol{d})+\frac{1}{2}\left({}^k\boldsymbol{\sigma}\right)\cdot\cdot\left[(\delta\boldsymbol{h})^T\cdot\left({}^k\boldsymbol{h}\right)+\left({}^k\boldsymbol{h}\right)^T\cdot(\delta\boldsymbol{h})\right]-$$

$$-\left[{}^k\Delta_y\cdot{}^k\mathbf{v}\right]{}^k\boldsymbol{f}\cdot\delta\mathbf{v}\}d\Omega_k+\int_{S_k^\sigma}\left\{{}^k\boldsymbol{t}_n\cdot\left({}^k\boldsymbol{h}\right)^T-\left[{}^k\Delta_y\cdot{}^k\mathbf{v}\right]^k\boldsymbol{t}_n\right\}\delta\mathbf{v}dS_k=$$

$$=\int_{S_k^\sigma}{}^k\vec{\boldsymbol{t}}_n\cdot\delta\vec{\mathbf{v}}dS_k+\int_{\Omega_k}{}^k\vec{\boldsymbol{f}}\cdot\delta\vec{\mathbf{v}}d\Omega_k-\frac{1}{\Delta t}\left\{\int_{\Omega_k}\left({}^k\boldsymbol{\sigma}\right)\cdot\cdot\left(\delta^k\boldsymbol{d}\right)d\Omega_k-\int_{S_k^\sigma}{}^k\vec{\boldsymbol{t}}_n\cdot\delta\vec{\mathbf{v}}dS_k-\int_{\Omega_k}{}^k\vec{\boldsymbol{f}}\cdot\delta\vec{\mathbf{v}}d\Omega_k\right\}.$$

For static problems take ${}^k\boldsymbol{v}=\frac{\Delta^k\boldsymbol{u}}{\Delta t}$. Since the argument t is arbitrary, allowable to take $\Delta t=1$. As a result, we get the resolving equation for the displacement increments $\Delta^k\vec{\boldsymbol{u}}$.

4. Numerical example

The numerical implementation is based on the finite element method [14]. An 8-node brick element is used. As an example the strain energy density of the neo-Hookean material is considered:

$$W=\frac{\mu}{2}\left(tr(\boldsymbol{B})-3\right),$$

where μ – shear modulus.

Elasticity tensor $\mathbf{c}$ is written as a sum of three terms:

$$\mathbf{c} = \hat{\mathbf{c}} + \mathbf{c_p} + \mathbf{c_k},$$

where

$$\hat{\mathbf{c}} = 2\mu J^{-\frac{5}{3}}\left(-\frac{1}{3}(\boldsymbol{B})\cdot(\boldsymbol{I}) + \frac{1}{9}I_{1\boldsymbol{B}}\cdot(\boldsymbol{I})\cdot(\boldsymbol{I}) + \frac{1}{3}(\boldsymbol{C}_{II})\cdot I_{1\boldsymbol{B}} - \frac{1}{3}(\boldsymbol{I})\cdot(\boldsymbol{B})\right),$$

$$\mathbf{c_p} = pJ\left((\boldsymbol{I})\cdot(\boldsymbol{I}) - 2(\boldsymbol{C}_{II})\right).$$

Here we have introduced the tensor of the fourth rank $(\boldsymbol{C}_{II}) = (\vec{e}_i\vec{e}_j\vec{e}_i\vec{e}_j)$.

The Cauchy-Euler stress tensor then takes the following form:

$$(\boldsymbol{\sigma}) = \frac{2}{J}(\boldsymbol{B})\cdot\left(\frac{\partial W}{\partial \boldsymbol{B}}\right) = \mu J^{-\frac{5}{3}}\left((\boldsymbol{B}) - \frac{1}{3}(\boldsymbol{I})\cdot I_{1\boldsymbol{B}}\right) + p(\boldsymbol{I}).$$

The plane stress of the strip 6.5×6.5×0.079 mm^3 with a hole 0.5 mm in diameter is considered [5]. We use neo-Hookean material with μ=0.4225 N/mm^2.The strip was stretched in the horizontal direction in six time (Figure 1). In the Figure 2 the finite element model is shown and in the Figure 3 – the final mesh.

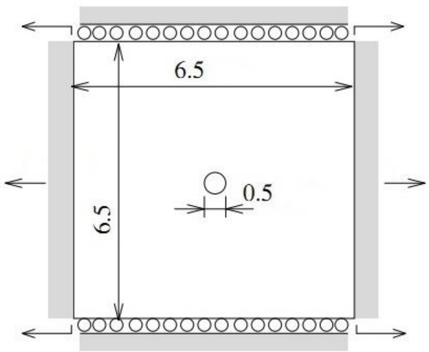

Figure 1. Strip with hole.

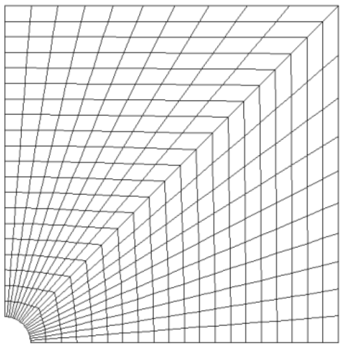

Figure 2. Finite element model.

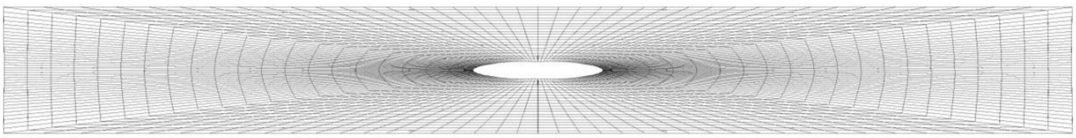

Figure 3. Final mesh.

5. Conclusion

Thus, in the paper we construct a method of numerical investigation of solids that physical relationships are specified using the strain energy density. The constitutive equations and resolving equation are obtained. The problem of plane deformation of the strip is solved. The calculation results show efficiency of this method of investigation of nonlinear problems.

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