

Application of a second order accurate finite-difference method to problems of diffraction of elastic waves by gradient layers

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Abstract. A generalized statement is formulated for the boundary-value problem describing diffraction of elastic waves by gradient isotropic and transversely isotropic layers. Numerical experiments are conducted for various types of materials filling the layer. A conclusion is drawn that the obtained finite-difference scheme is second-order accurate, when distributions of elastic parameters in the layer are described by smooth curves.

1. Introduction

Boundary value problems for the Lamé equations are encountered in consideration of processes of propagation and diffraction of elastic waves. Passage of waves through media with continuously changing parameters represent a special interest here. One of the simplest approaches to analyzing diffraction of an elastic wave by layers of a varying structure is applying an isotropic model of the layer. A more general approach to studying layers of a varying structure is through applying a transversely isotropic model of the layer. In our previous studies, we investigated a problem of passage of a planar elastic wave through gradient isotropic [1], transversely isotropic [2] and anisotropic layers [4]. A disadvantage of those studies was that the used numerical methods were first-order accurate.

As an alternative method of solution of problems of this type, one can consider the finite-element method being widely used in recent years [5], which is particularly efficient for solving multidimensional problems in domains with complex boundaries [6]. However, in the one-dimensional case finite-difference methods can be competitive to the finite-element methods. One of such approaches, called the method of approximation of integral identities [7], is used in the present study.

The method of approximation of integral identities allows achieving the approximation error $O(h^2)$. This algorithm of constructing finite-difference schemes was applied by us earlier to solving the Helmholtz equation, to which a one-dimensional problem of diffraction of an elastic wave by a gradient layer is reduced [8], as well as to solving the saturated-unsaturated filtration consolidation problem [9].

In the present study, a concept of a generalized solution to the Cauchy problem describing the process of diffraction of elastic waves by gradient isotropic and transversely isotropic layers is



introduced and its uniqueness conditions are formulated. A second-order accurate finite-difference scheme for solving the problem is presented. A series of numerical experiments demonstrating applicability of the method in terms of smoothness of curves of distribution of the layer's elastic parameters is provided.

2. Problem statement

Let an elastic harmonic wave u_0 fall on a uniform in the transverse direction and transversely isotropic layer of thickness L (medium 2 $\{0 < x < L\}$ having continuous density $\rho_2(x)$ and tensor (3×3) of elasticity modulus $K(x)$ from medium 1 $\{x > L\}$ at angle θ with respect to axis x (see Fig. 1). As a result of diffraction, there appear a wave reflected to medium 1, a wave passing to medium 3 $\{x < 0\}$ and a field in the layer. We will assume that the media 1 and 3 are uniform and isotropic.

The process of diffraction of a planar elastic wave by a gradient isotropic and transversely isotropic layer (at $x \in (0, L)$) is governed by the system of ordinary differential equations of the second order with respect to displacement components in the Cartesian coordinate system $u = (u_1(x), u_2(x))$, which has the form:

$$\frac{d}{dx} \left(c_{s1} \frac{du_s}{dx} \right) + c_{s2} u_s + c_{s3} \frac{du_k}{dx} + c_{s4} u_k = q_s, \quad s, k = 1, 2, \quad s \neq k, \quad (1)$$

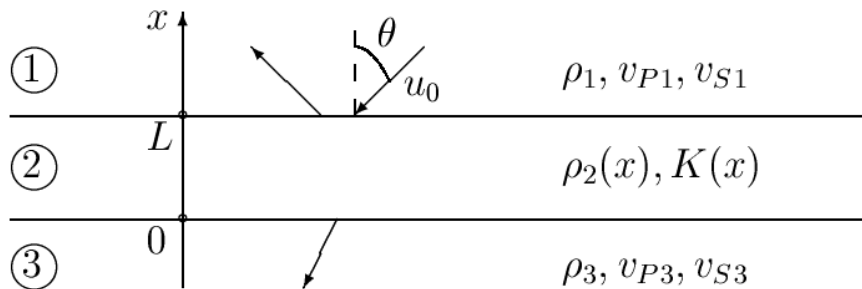


Figure 1. Geometry of the problem.

and boundary conditions have the form ($s, k = 1, 2, \quad s \neq k$):

$$a_{s1} \frac{du_s(0)}{dx} + a_{s2} u_s(0) + a_{s3} \frac{du_k(0)}{dx} + a_{s4} u_k(0) = f_s, \quad (2)$$

$$b_{s1} \frac{du_s(L)}{dx} + b_{s2} u_s(L) + b_{s3} \frac{du_k(L)}{dx} + b_{s4} u_k(L) = g_s. \quad (3)$$

Expressions (2), (3) correspond to conditions of continuity on boundaries $x=0$ and $x=L$ of normal and tangential stresses as well as of both displacement components.

We will assume that functions $c_{si}(x)$, $q_s(x)$, $i=1..4$ and $s=1,2$ are continuous complex-valued functions on interval $[0, L]$, and the following relations are fulfilled:

$$c_{s1}(0) \neq 0, \quad c_{s1}(L) \neq 0, \quad s=1,2, \quad (4)$$

and the numbers a_{si} , b_{si} , $i=1,2,3$, $s=1,2$ satisfy the conditions:

$$a_{11} \cdot a_{21} - a_{23} \cdot a_{13} \neq 0, \quad b_{11} \cdot b_{21} - b_{23} \cdot b_{13} \neq 0. \quad (5)$$

Equation (4) has the following physical meaning. If the coefficients $c_{s1}(x)$ are not equal to zero, then it means that longitudinal and transverse velocities at points $x = 0$ and $x = L$ are also not equal to zero for the case of equation (1) describing propagation of an elastic wave in the gradient isotropic layer. In addition, the latter means that, for the case of transversely isotropic media, the longitudinal and transverse velocities in the isotropy plane are not equal to zero either [3].

With regard to conditions (5), it is worth noting that if the boundary-value problem (1)–(3) describes diffraction by gradient layers, then conditions (5) imply lack of excitation of surface modes (Rayleigh waves) in the adjacent half planes.

Let us construct a second-order accurate finite-difference scheme for solving the problem (1)–(3) and estimate the error of the obtained solution at various distributions of the layer's elastic parameters. At first, we will proceed from the classical solution to the generalized solution.

3. Generalized solution to the problem

If conditions (4), (5) are fulfilled, then system (1)–(3) can be rewritten in the following equivalent operator form [10]:

$$-\frac{d}{dx}\left(A\frac{du}{dx}\right) + B\frac{du}{dx} + Du = q, \quad x \in (0, L), \quad (6)$$

$$\left(A\frac{du}{dx}\right)\Big|_{x=0} = G_0 u(0) + \varphi^0, \quad (7)$$

$$-\left(A\frac{du}{dx}\right)\Big|_{x=L} = G_L u(L) + \varphi^L, \quad (8)$$

where $u = (u_1(x), u_2(x))$ is a vector function; $\frac{du}{dx} = \left(\frac{du_1}{dx}, \frac{du_2}{dx}\right)$, A, B, D are linear operators acting from a two-dimensional complex plane C^2 into space C^2 ; $q = (q_1, q_2)$; $\varphi^0 = (\varphi_1^0, \varphi_2^0)$; $\varphi^L = (\varphi_1^L, \varphi_2^L)$; matrices G_0 and G_L are given in the form:

$$G_0 = \begin{pmatrix} \mu_{11}^0 & \mu_{12}^0 \\ \mu_{21}^0 & \mu_{22}^0 \end{pmatrix}, \quad G_L = \begin{pmatrix} \mu_{11}^L & \mu_{12}^L \\ \mu_{21}^L & \mu_{22}^L \end{pmatrix},$$

here complex numbers $\mu_{ij}^0, \mu_{ij}^L, \varphi_i^0, \varphi_i^L$ are determined via coefficients of boundary conditions a_{ik}, b_{ik}, f_i, g_i , where $i, j = 1, 2, k = 1..4$.

We define the space $V = W_2^{(1)}(0, L) \times W_2^{(1)}(0, L)$ having the norm

$$\|u\| = \int_0^L \left(|u|^2 + \left| \frac{du}{dx} \right|^2 \right) dx$$

and introduce the concept of a generalized solution to problem (6)–(8) in the form of a function $u = (u_1(x), u_2(x)) \in V$. For function u , the following equality holds:

$$\begin{aligned} \int_0^L \left(A \frac{du}{dx} \cdot \frac{dv}{dx} + B \frac{du}{dx} \cdot v + Du \cdot v \right) dx + G_0 u(0) \cdot v(0) + G_L u(L) \cdot v(L) = \\ = \int_0^L q \cdot v dx - \varphi^0 \cdot v(0) - \varphi^L \cdot v(L) \quad \forall v = (v_1, v_2) \in V, \end{aligned} \quad (9)$$

where $u \cdot v = u_1 \bar{v}_1 + u_2 \bar{v}_2$ is a standard scalar product in C^2 .

Next, we define sesquilinear form $a: V \times V \rightarrow C$ and linear functional $f: V \rightarrow C$ using the following relations:

$$a(u, v) \equiv \int_0^L \left(A \frac{du}{dx} \cdot \frac{dv}{dx} + B \frac{du}{dx} \cdot v + Du \cdot v \right) dx + G_0 u(0) \cdot v(0) + G_L u(L) \cdot v(L), \quad (10)$$

$$f(v) \equiv \int_0^L q \cdot v dx - \varphi^0 \cdot v(0) - \varphi^L \cdot v(L). \quad (11)$$

Then, seeking a generalized solution to problem (6)–(8) is equivalent to seeking a function $u \in V$, meeting the condition:

$$a(u, v) = f(v) \quad \forall v \in V. \quad (12)$$

We set the following conditions for functions c_{ij} , q_i and matrices G_0, G_L :

A_1) Functions $c_{ij} \in L_\infty(0, L)$, $q_i \in L_2(0, L)$, $i = 1, 2$, $j = 1..4$.

A_2) $\operatorname{Re}(c_{i1}(x)) \geq \alpha_{i1}$, $x \in [0, L]$, $i = 1, 2$.

A_3) $\operatorname{Re}(G_0 \xi \cdot \xi) \geq \alpha_0 |\xi|^2$, $\operatorname{Re}(G_L \xi \cdot \xi) \geq \alpha_L |\xi|^2 \quad \forall \xi \in C^2$, where α_{11} , α_{21} , α_0 , α_L are some positive constants.

We will assume that the conditions $A_1) - A_3)$ are met along with a condition providing V -ellipticity of form a , i.e. there exists a positive real number δ_3 such that for any $u \in V$, the following inequality is satisfied:

$$\operatorname{Re}(a(u, u)) \geq \delta_3 \|u\|. \quad (13)$$

Then, problem (6)–(8) has a unique generalized solution $u \in V$ and the following inequality is true:

$$\|u\| \leq c \cdot \left(\sum_{i=0}^2 (\|q_i\|_{L_2} + |\varphi_i^0| + |\varphi_i^L|) \right), \quad (14)$$

where c is a positive constant.

4. Approximation of the problem

For approximating the problem (6)–(8), we will use the method of approximation of integral identities. For that, on interval $[0, L]$ we will construct a uniform grid $\bar{\omega}$ with step size $h = L/N$:

$$\bar{\omega} = \{x_i = ih, \quad i = 0..N\}.$$

Using a classical approach [7], we will define a space of grid functions V_h on $\bar{\omega}$.

We call grid functions $(y_1, y_2) \in V_h$ as a solution to the finite-difference scheme of problem (6)–(8), if, for any grid functions $\eta_1, \eta_2 \in V_h$ and $s, k = 1, 2$, $s \neq k$, the following inequalities are satisfied [10]:

$$\begin{aligned} & -\frac{1}{2} [c_{s1}(y_s)_x, (\eta_s)_x] - \frac{1}{2} (c_{s1}(y_s)_{\bar{x}}, (\eta_s)_{\bar{x}}) \\ & + \frac{1}{2} [c_{s3}(y_k)_x, \eta_s] + \frac{1}{2} (c_{s3}(y_k)_{\bar{x}}, \eta_s) + [c_{s2}y_s + c_{s4}y_k, \eta_s] \\ & + (\mu_{s1}^0 y_{1,0} + \mu_{s2}^0 y_{2,0} + \varphi_s^0) \bar{h}_{s,0} + (\mu_{s1}^L y_{1,N} + \mu_{s2}^L y_{2,N} + \varphi_s^L) \bar{h}_{s,N} = [q_s, \eta_s]. \end{aligned} \quad (15)$$

Let (u_1, u_2) be a solution to problem (6)–(8), and functions c_{s1} and u_s have three and four constrained derivatives in the neighborhood of nodes of grid $\bar{\omega}$, respectively. Then, the finite-difference scheme (15) has approximation error $\Psi_i^s = O(h^2)$, $s = 1, 2$, $i = 0..N$.

It is worth noting that the finite-difference scheme (15) represents a linear system of equations, for which one can apply the Thomas algorithm [7].

5. Numerical results

Let us estimate an error of solution of problem (1)–(3) by using the finite-difference scheme (15). For that purpose, we will consider some model problems having known solutions (u_1, u_2) . We calculate functions $c_{si}(x)$ and coefficients a_{si}, b_{si} for certain materials filling the layer. By substitution of known functions (u_1, u_2) into (1)–(3), we find right-hand sides $q_s(x)$, f_s and g_s . Next, we solve numerically the Cauchy problem with coefficients and right-hand sides obtained. As a results, will obtain an exact (a-priori known) solution $u = (u_1, u_2)$ and approximated solution $y = (y_1, y_2)$. For determining the error of the obtained solution $(y_{1,j}, y_{2,j})$ in node x_j ($j = 0..N$), we calculate the norm:

$$\|y_j - u(x_j)\| = \left(|y_{1,j} - u_1(x_j)|^2 + |y_{2,j} - u_2(x_j)|^2 \right)^{1/2}. \quad (16)$$

An error of the approximated solution in the entire layer will be determined through the maximum norm:

$$\varepsilon = \max_{j=0..N} \|y_j - u(x_j)\|. \quad (17)$$

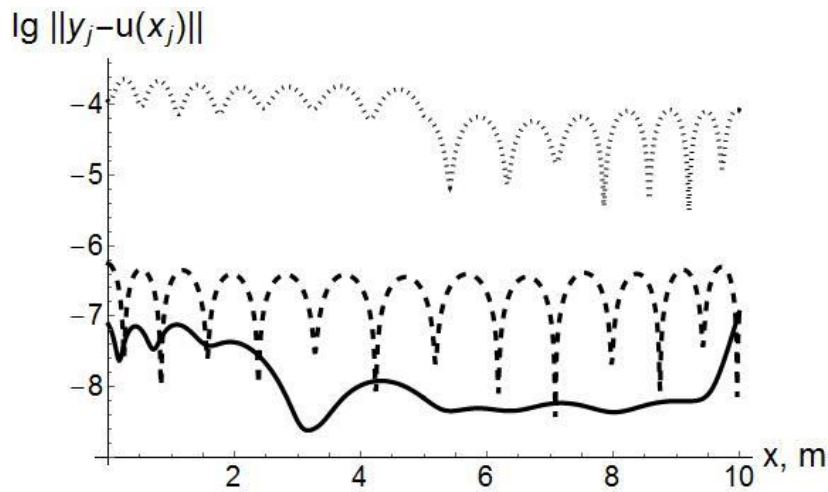


Figure 2. Values of logarithm of the error of the approximated solution in the layer's points. Solid line corresponds to a linear change of elastic parameters; dashed line corresponds to a parabolic profile; dotted line corresponds to a broken profile.

Let us consider a layer with transversely isotropic material located inside a sand-rock and having parameters: $\rho_1 = \rho_3 = 2400 \text{ kg/m}^3$; $v_{P1} = v_{P3} = 3300 \text{ m/s}$ and $v_{S1} = v_{S3} = 2200 \text{ m/s}$. We confine ourselves only to three numerical experiments. The first numerical experiment corresponds to the case of the layer's elastic parameters changing continuously from sand-rock (isotropic material) to silt-rock (transversely isotropic material). At $x = 0$, we have a continuous transition from the external medium toward the layer. After that, the layer parameters as well as coefficients of the system of equations (1) also change continuously, obeying the linear law, toward parameters corresponding to a transversely isotropic material, i.e. silt-rock [11]: $E = 56.8 \text{ GPa}$, $E' = 62.1 \text{ GPa}$, $\nu = 0.29$, $\nu' = 0.26$, $G' = 22.9 \text{ GPa}$ (more details on connections between elastic parameters for a transversely isotropic

material and elasticity tensor K can be found in [2]). At another point of contact of the layer with sand-rock, $x = 10\text{ m}$, parameters change discontinuously, by a leap.

The second numerical experiment is carried out for the case of elastic parameters changing continuously by the parabolic law from sand-rock at the beginning of the layer toward silt-rock at $x = 5\text{ m}$ followed by a symmetric change of parameters back toward sand-rock at $x = 10\text{ m}$. The third numerical experiment is carried out for the case of elastic parameters changing continuously and linearly from sand-rock toward silt-rock in the middle of the layer followed by a continuous and linear change toward sand-rock by the end of the layer. In the latter case, for elastic parameters we have a broken profile, i.e. a profile having discontinuity of the first derivative in the middle of the layer. We assume constant density throughout the layer for all considered cases: $\rho_2 = 2400\text{ kg/m}^3$.

System (1)–(3) is solved for the problem of a planar longitudinal wave of a single amplitude falling on the layer at angle $\theta = 20^\circ$.

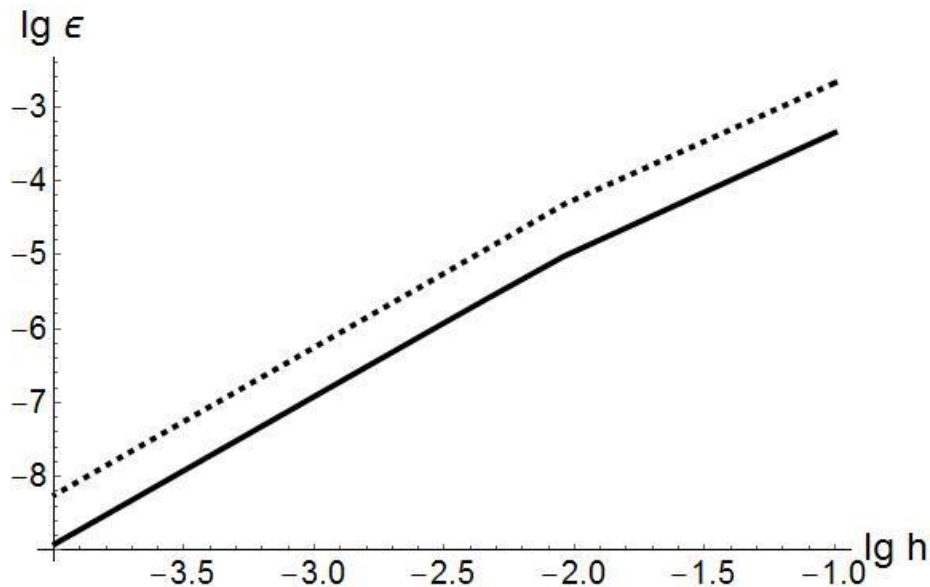


Figure 3. Dependence of logarithm of the approximated solution on logarithm of the grid step size. Solid line corresponds to a linear profile; dotted line corresponds to a parabolic profile.

Let us estimate an error of the approximated solution for each of the cases in the layer's points via formula (16) choosing the grid step size $h = 10^{-3}$. An error for the linear profile, when compared with the other experiments, is smallest (solid line in Fig. 2). At the end point, there is slight worsening of the error, which is explained with discontinuity of elastic parameters at that point. The error for the parabolic profile (dashed line) is slightly worse, but it is still of order $O(h^2)$. In case of the broken profile (dotted line), the error becomes of the first order due to breakdown of smoothness of coefficients of the original equation (1) in the central point $x = 5\text{ m}$.

Now we will consider dependence of the solution error in the entire layer ε (17) on the grid step size. One can see from Fig. 3 that, for experiments with smooth coefficients of the original equation, logarithm of the error is proportional to logarithm of the error. The solution error for the linear profile is smaller than that for the parabolic profile.

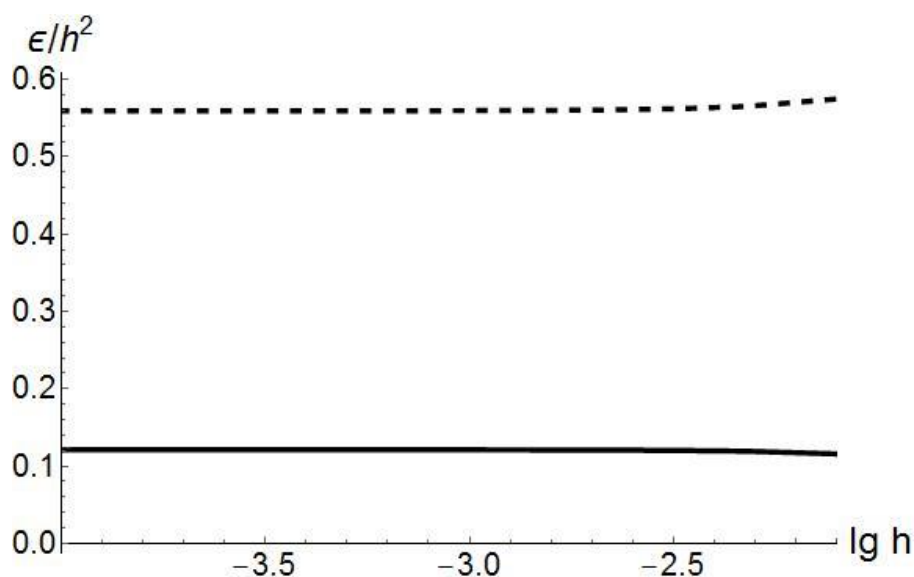


Figure 4. Graphs depicting values of the constant C in estimating $\varepsilon \leq Ch^2$. Solid line corresponds to the linear profile; dashed line corresponds to the parabolic profile.

Thus, one can draw a conclusion that the numerical scheme has second-order accuracy for problems with smooth coefficients. For clarifying the coefficient in estimating $\varepsilon \leq Ch^2$, let us present graphs for dependence of ε/h^2 on $\lg h$. We will obtain that $C \approx 0.12$ for the linear profile case and $C \approx 0.56$ for parabolic profile case. This also confirms our assumption that the problem with linear profile is solved more accurately. Note that the obtained estimations for C are valid at $h < 0.01$, and at $h > 0.01$, the values of C slightly increase.

6. Conclusions

A conclusion is drawn that for the problem of diffraction of an elastic wave by a gradient layer with elastic parameters having smooth distributions, the finite-difference scheme obtained by the method of approximation of integral identities is second-order accurate. It is shown that in case of breakdown of smoothness of curves of distribution of elastic parameters in the layer, the error of the scheme reduces to the first-order.

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