Vector random fields in mathematical modelling of electron motion

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Abstract. In order to use electron motion for mathematical modelling we assessed convergence rate in central limit theorem for homogeneous vector random fields, which fulfil condition of strong mixing.

1. Introduction

Capacitive coupled radio-frequency (CCRF) discharges are widely used for treatment of organic materials because they have low gas temperature. When making numerical calculation of CCRF-discharge characteristics in approximation of continuous medium, we need apply the values of the factors of the initial-boundary value problems, which are the part of the mathematical model of the discharge (factors of diffusion, mobility of charged factors, velocity of plasma chemical reaction) [1-3]. We have very little experimental information, that is why the factors for electron gas are determined by solving Boltzmann equation, and Maxwellian distribution function is used for ion gas as a rule. However, these methods are not accurate under high values of the electrical field observed in electrode sheath of glow discharge and CCRF discharge; in order to account impact of the field, they apply statistical calculation methods for velocity distribution function of discharged particles [4-7]. Here we use statistical analysis of random fields. It is known that limit theorems are significantly important for statistical methods. Many authors use central limit theorem for random fields under some low dependency limits [7-10]. This research paper is dedicated to application of asymptotic decomposition of the characteristic functions and assessment of sum semi-invariants of random vectors in order to obtain values of convergence rate in limit theorem for weakly dependent vector random fields.

2. Statement of a theorem

Let us assume that $\xi(t_1,...,t_n)$ – homogenous, in restricted sense, vector random field, assumed on integer lattice Z^n, in other words the random function reflecting Z^n in m-dimensional Euclidean space R^m.

We shall set $S_{T_1,...,T_n} = \sum_{t_1=1}^{T_1} ... \sum_{t_n=1}^{T_n} \xi(t_1,...,t_n)$, $d(E_1,E_2) = \inf_{t\in E_1, s\in E_2} \sqrt{\sum_{i=1}^{n}(t_i - s_i)^2}$, $t=(t_1,...,t_n)$, $s=(s_1,...,s_n)$.

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We shall introduce the function $\alpha(E_1,E_2)=\sup|P(AB)-P(A)P(B)|$ for any invariant sets E_1,E_2 from Z^n, where the upper boundary is taken at all events A,B belonging to minimal σ-algebras, resulted by $\xi(t_1,...,t_n)$, when the vector $(t_1,...,t_n)$ belongs to E_1,E_2, correspondingly.

Theorem. Let us assume that the following conditions are applied:

1) Random vectors $\xi(t_1,...,t_n)$ have zero vector of mathematical mean value;

2) $|\xi(t_1,...,t_n)|\le B$, B- positive constant;

3) random field $\xi(t_1,...,t_n)$ corresponds to the condition of strong mixing with the factor $\alpha(E_1,E_2)\le Ad(E_1,E_2)^{-\omega}$,where $E_1\cap E_2=\varnothing$, ω- significantly large fixed positive number, A- positive constant;

4) covariance matrix of the sum $S_{T_1},...,_{T_n}/\sqrt{T_1T_2...T_n}$ is not confluent and under $T_1,...,T_n\to\infty$ is approaching to the unit matrix.

So, $\sup\limits_M \left|P(S_{T_1,...,T_n}/\sqrt{T_1...T_n}\in M)-\Phi(M)\right|=O((T_1...T_n)^{-\frac{1}{2n}+\varepsilon})$, under $T_1,...,T_n\to\infty$, where the upper boundary is taken at all convex measurable sets $M\subset R^m$, $\varepsilon=3n/((n-1)\sqrt[8]{\omega})$, Φ- normal distribution with unit matrix of co-variations and zero mean vector

3. Theorem proof.

The theorem is proved by using “step-by-step approximations” method developed by V.T. Dubrovin, D.A. Moskvin to study rates of convergence in boundary value theorems for the sum functions from slightly dependent values; development of this method was continued by F.G. Gabbasov for multidimensional boundary value theorems for slightly dependent vectors [11-15].

Let us provide the values used to prove the theorem. We shall introduce a random value, where $x=(x_1,...,x_m)$, $|x|=\sqrt{\sum\limits_{i=1}^m x_i^2}$. We shall specify $\chi_\nu(Q_1,...,Q_n)$ a semi-invariant of ν- order of the sum $\sum\limits_{k_1=1}^{Q_1}...\sum\limits_{k_n=1}^{Q_n}\tau_{k_1,...,k_n}$.

Lemma 1. The following estimate is fair

$$|\chi_\nu(Q_1,...,Q_n)|=O(Q_1Q_2...Q_n).$$

The lemma is proved with the same way as in [12].

Lemma 2. Under fixed ν, $1\le\nu<\omega/n$, the estimate is fair

$$E\left|\sum_{k_1=1}^{h_1}...\sum_{k_n=1}^{h_n}\tau_{k_1...k_n}\right|^{2\nu}=O((h_1h_2...h_n)^{\nu+2n\nu^2/\omega}$$

The lemma shall be proved as lemma 2 from [12].

Let us assume that P_Q-distribution of the sum $\sum\limits_{k_1=1}^{Q_1}...\sum\limits_{k_n=1}^{Q_n}\xi(k_1,...,k_n)$, and $f_Q(x)$-its characteristic function.

Lemma 3. If under some $0<\alpha\le 1/2$, and some $\gamma,\gamma>1$ asymptotic equality takes place

$$|P_Q(M)-\Phi(M)|=O(\frac{1}{1+\beta^\gamma(M)}\ \frac{1}{(Q_1...Q_n)^\alpha}) \qquad (1)$$

under $Q_1,...,Q_n\to\infty$, for every $\delta>0$ there are such $\Delta,\varepsilon>0$, the inequation $\max\limits_{\delta<|x|<\varepsilon(Q_1...Q_n)^\alpha}|f_Q(x)|\le 1-\Delta$ holds, $\beta(M)=\inf\limits_{x\in\delta(M)}|x|$, where $\delta(M)$ is boundary of convex set M.

The lemma shall be proved with the similar way as lemma 2 from the research paper [13]

Let us introduce the following symbols. Let us assume that N and Q_i are random, growing together with $T_1,...,T_n$, natural numbers. Let us assume that

$$E_{k_1,...,k_n=}\{(t_1,...,t_n)\in Z^n:(k_i-1)(Q_i+N)<t_i\le k_i(Q_i+N),i=1,n\},$$

$$E'_{k_1,...k_n}=\{(t_1,...,t_n)\in Z^n:(k_i-1)(Q_i+N)<t_i\le(k_i-1)(Q_i+N)+Q_i,i=1,n\}$$

$$E^0_{p_1+1,...,p_n+1}=\{(t_1,...,t_n)\in Z^n:p_i(Q_i+N)<t_i\le T_i,i=1,n\},$$

$$E^0_{k_1,...,k_n}=E_{k_1,...,k_n}/E'_{k_1,...,k_n},$$

where $1\le k_i\le p_i$, $p_i=[T_i/(Q_i+N)]$, [ ] – the sign of an integral part. Moreover, let us assume, that

$$|T_i-p_i(Q_i+N)|\le p_i, i=\overline{1,n}. \tag{2}$$

Let us indicate $Q=Q_1Q_2...Q_n$, $p=p_1p_2...p_n$.

Further, let us compose the sums

$\eta_{k_1,...,k_n=}S_{E'_{k_1,...,k_n}}/\sqrt{Q}$, $\eta^0_{k_1,...,k_n}=S_{E^0_{k_1,...,k_n}}/\sqrt{Q}$, $\eta^0_{p_1+1,...,p_n+1}=S_{E^0_{p_1+1,...,p_n+1}}/\sqrt{Q}$ where summing $\xi(t_1,...,t_n)$ is made according to the stated subsets of integral lattice Z^n. We shall compose the sums

$$\varsigma_p=\sum_{k_1=1}^{p_1}...\sum_{k_n=1}^{p_n}\eta_{k_1,...,k_n},\ \varsigma^0_p=\sum_{k_1=1}^{p_1+1}...\sum_{k_n=1}^{p_n+1}\eta^0_{k_1,...,k_n}.$$

Now the sum $S_{T_1,...T_n}=\varsigma_p\sqrt{Q}+\varsigma^0_p\sqrt{Q}$. Here, due to the condition of strong mixing 3) theorems, random values in the sum ς_p are almost independent and we can apply the central limit theorem for the sums of independent random vectors including evaluation of convergence rate. The sum ς^0_p does not affect general limit distribution and its contribution of distribution of the sum $S_{T_1,...T_n}$ passes to a residual member of the theorem.

Let us demonstrate it. We shall indicate $\hat{\eta}_{k_1,...,k_n}$ random vectors, which are distributed by the same way as $\eta_{1,...,1}$ and $\hat{\varsigma}_p=\sum_{k_1=1}^{p_1}...\sum_{k_n=1}^{p_n}\hat{\eta}_{k_1,...,k_n}$.

Let us assume that Λ is the matrix of vector covariate $\eta_{1,...,1}$. Taking into account condition 4) of the theorem, we can show that the elements Λ differ from the elements of unit matrix at the value $O(1/Q)$. Let us indicate A such matrix, that $A'A=\Lambda^{-1}$, where A' - is transposed matrix A. It is obviously that the vector $A\hat{\varsigma}_p/\sqrt{p}$ has unit matrix of covariate.

Let us assume that G_p - is distribution of $\varsigma_p/\sqrt{p}$; G^A_p - is distribution of $A\varsigma_p/\sqrt{p}$; $f_p(x)$, $\hat{f}_p(x)$ - are characteristic functions of $A\varsigma_p/\sqrt{p}$, $A\hat{\varsigma}_p/\sqrt{p}$, respectively.

By use of condition of strong mixing 3) of the theorem, we can get (see.[13], balance(6))

$$\left|f_p(x)-\hat{f}_p(x)\right|=O(p/N^{\sqrt[4]{\omega^3}}).$$

More over, under $|x|\le\sqrt{p}/(8(E|A\eta_{1,...,1}|^{\nu+1})^{1/(\nu+1)})=T_{\nu p}$ is right (see [13], ratios (7) and (8)) asymptotic decomposition

$$\hat{f}_p(x)=\exp(-|x|^2/2)(1+\sum_{r=1}^{\nu}P_r(ix)(1/p^{r/2}))+O(3^{\nu+2}|x|^{\nu+3}\exp(-|x|^2/4)/T^{\nu+1}_{\nu p}),$$

where $P_r(ix)$ are dependent on semi-invariants of the value $(A\eta_{1,1,...1},ix)$ of the order not higher than r+2 (see [13] of ratios (7) and (8)). Lemma 1 implies that $|P_r(ix)|=O(r^{2r}|x|^{r+2}/Q^{r/2})$.

In further we use S.M. Sadikova's inequation – linking difference between the characteristic functions with the difference between corresponding distributions. The methods of its use for this specific case

is described in [13], where we shall choose $T = O(Q^{\alpha}\ T_{vp})$ and consider that the conditions of lemma 3 are fulfilled.
Thus we get,

$$\sup_M \left|G_p^A(M) - \Phi(M)\right| = O(\ln^{m/4}(T)(pT^m / N^{\sqrt[4]{\omega}} + 1/\sqrt{pQ} + 1/T_{vp}^{\nu+1} + Q^{m\alpha} p^{m/2} \exp(-cp) + \ln T / T))$$

In further, as well as in [13] we can show that evaluation of $\sup_M \left|G_p(M) - \Phi(M)\right|$ differs from preliminary evaluation on $O(1/Q)$. Than after evaluation of distribution $G_p(M) = P(\varsigma_p / \sqrt{p} \in M)$ we start evaluating distribution $P((\varsigma_p + \varsigma_p^0)/\sqrt{p} \in M) = P(1/\sqrt{pQ}\sum_{k_1=1}^{T_1}\ldots\sum_{k_n=1}^{T_n}\xi(k_1,\ldots,k_n) \in M)$ as well as in [13], by using lemma 2. Thus, we get order difference $O((N+1)/Q^{1/(2n)})$. The reason of such rough evaluation is in large amount of summands in ς_p^0 and in the applied method, which can be improved in future. Similarly, we start evaluating distribution $P(1/\sqrt{n}\sum_{j=1}^{n}\xi_j \in M)$ and get order difference $O((T_1 \ldots T_n)^{\nu+2n\nu^2/\omega} / (Q^{2\nu} p^{\nu}))$. Than due to choice of $\nu = \sqrt[3]{\omega}, N = (T_1 \ldots T_n)^{1/\sqrt[4]{\omega}}$, $p_i = [T_i^{(1-2n\alpha)/(n+1-2n\alpha)}]$, and Q_i from the condition (2) we get

$$\left|P(S_{T_1,\ldots,T_n} / \sqrt{T_1 \ldots T_n} \in M) - \Phi(M)\right| = O((T_1 \ldots T_n)^{\frac{1}{2(n+1-2n\alpha)} + \frac{1}{\sqrt[4]{\omega}}})$$

Thus, the evaluation (see [13], lemma 5) follows:

$$\left|P(S_{T_1,\ldots,T_n} / \sqrt{T_1 \ldots T_n} \in M) - \Phi(M)\right| = O(1 \big/ ((1+\beta^{\sqrt[12]{\omega}})(T_1 \ldots T_n)^{\frac{1}{2(n+1-2n\alpha)} - \frac{2}{\sqrt[4]{\omega}}}))$$

This evaluation is obtained in presumption that condition (1) of lemma 3 is fulfilled. If we assume that α =0, so we get

$$\left|P(S_{T_1,\ldots,T_n} / \sqrt{T_1 \ldots T_n} \in M) - \Phi(M)\right| = O(1 \big/ ((1+\beta^{\sqrt[12]{\omega}})(T_1 \ldots T_n)^{\frac{1}{2(n+1)} - \frac{2}{\sqrt[4]{\omega}}})) \quad (3)$$

without using lemma 3. This result can be considered as the initial approximation for our method. Due to use of condition (1) of lemma 3 under (3) is fulfilled, we get more precise residual member in the theorem.

As a result of step-by-step approximation, we get proof for our theorem as well as in [12]. We shall notice that condition 4) of unicity of covariance matrix does not limit the community as the distance between distribution of random vectors is invariant in relation to non-generate linear transformation of these vectors.

4. Conclusion

Thus, as a result of use of step-by-step approximation method we managed to distribute evaluation of convergence rate within boundary value theorem for uniform random fields one dimensional case to multidimensional case. In further there is possibility to improve this evaluation in order to reach the level of the boundary value theorems for independent random fields**.**

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References

[1] Chebakova V Ju, Zheltukhin V S and Dubrovin V T 2016 Problem of radio-frequency discharge at atmospheric pressure in local statement *Applied Mathematical Sciences* **10** (21) 1013-22

[2] Zheltuhin V S, Solovyev P S and Chebakova V Y 2015 Boundary conditions for electron balance equation in the stationary High-Frequency Induction discharges *Research Journal of Applied Sciences* **10** (10) 658-62

[3] Dubrovin V T, Chebakova V Ju and Zheltukhin V S. 2016 Radio-Frequency Discharge at Low Pressure: a Non-Local Problem Statement Approach *Procedia Engineering*. **150** 1041–5

[4] Demkin V P, Mel'nichuk S V, Borisov A V and Bardin S S 2013 Monte-Carlo calculation of the electron energy distribution function of a He–N2 plasma of a highvoltage pulsed discharge *Russian Physics Journal* **56** (4) 486-8

[5] Hans Rau 2000 Monte Carlo simulation of a microwave plasma in hydrogen *J. Phys. D: Appl. Phys*. **33** 3214–22

[6] Yousfi M, Hennad A and Alkaa A 1994 Monte Carlo simulation of electron swarms at low reduced electric fields *Physical review E* **49**(4) 3264-73

[7] Bulinskiy A V 1979 Central limit theorem for the random fields with strong and slight dependence. *The Paper of Academy of Science of USSR* **248** (1) 17-9 [in Russian]

[8] Bulinski A V and Shashkin A P 2004 Rates in the central limit theorem for the sums of dependent multiindexed random vectors *Journal of Math. Sciences* **122** (4) 3343-58

[9] Bulinskiy A V and Shashkin A P 2008 *Boundary value theorems for associated random fields and related systems* (Moscow: PHYSMATHLIT). [in Russian]

[10] Zuev N M 2000 *Boundary value theorems for slightly dependent random values* (Mn: BGU) [in Russian]

[11] Dubrovin V T, Gabbasov F G, Chebakova V J 2016 Multidimensional central limit theorem for sums of functions of the trajectories of endomorphisms *Lobachevskii Journal of Mathematics*, **37**(4) 409-17

[12] Aminev F A and Dubrovin V T 1989 A rate of convergence bound in the central limit theorem for weakly dependent homogeneous random fields *Journal of Soviet Mathematics* **44** (5) 559-67

[13] Gabbasov F G 1977 A multidimensional central limit theorem for sums of functions of sequences with mixing *Lithuanian Mathematical Journal* **17** (4) 494-505

[14] Gabbasov F G and Dubrovin V T 1988 Probability that the normed sum of weakly dependent vectors falls in the exterior of a convex set *Journal of Soviet Mathematics* **43** (1) 2187-93.

[15] Dubrovin V T 2014 Convergence rate in limit theorems for weakly dependent random values *Lobachevskii Journal of Mathematics* **35**(4) 390-6