On the existence of solutions of one nonlinear boundary-value problem for shallow shells of Timoshenko type with simply supported edges

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Abstract. Solvability of one system of nonlinear second order partial differential equations with given initial conditions is considered in an arbitrary field. Reduction of the initial system of equations to one nonlinear operator equation is used to study the problem. The solvability is established with the use of the principle of contracting mappings. The method used in these studies is based on the integral representations for the displacements. These representations are constructed with the use of general solutions to the inhomogeneous Cauchy-Riemann equation.

1. Introduction

Let us introduce in the plane bounded domain Ω and consider a system of nonlinear differential equations in the form

$$\begin{gathered}
w_{1\alpha^1\alpha^1} + \mu_1 w_{1\alpha^2\alpha^2} + \mu_2 w_{2\alpha^1\alpha^2} = f_1, \\
\mu_1 w_{2\alpha^1\alpha^1} + w_{2\alpha^2\alpha^2} + \mu_2 w_{1\alpha^1\alpha^2} = f_2, \\
k^2\mu_1\left(w_{3\alpha^1\alpha^1} + w_{3\alpha^2\alpha^2} + \psi_{1\alpha^1} + \psi_{2\alpha^2}\right) + k_3 w_{1\alpha^1} + k_4 w_{2\alpha^2} - k_5 w_3 + \\
+ k_3 w_{3\alpha^1}^2/2 + k_4 w_{3\alpha^2}^2/2 + \beta_2\left[\left(T^{\lambda\mu} w_{3\alpha^\lambda}\right)_{\alpha^\mu} + R^3\right] = 0 \\
\psi_{1\alpha^1\alpha^1} + \mu_1\psi_{1\alpha^2\alpha^2} + \mu_2\psi_{2\alpha^1\alpha^2} = g_1 + k_0\psi_1, \\
\mu_1\psi_{2\alpha^1\alpha^1} + \psi_{2\alpha^2\alpha^2} + \mu_2\psi_{1\alpha^1\alpha^2} = g_2 + k_0\psi_2,
\end{gathered} \tag{1}$$

under the following conditions at the boundary Γ:

$$w_1 = w_3 = \psi_1 = 0, \tag{2}$$

$$\mu_1\left(w_{1\alpha^2} + w_{2\alpha^1}\right)(t)\,d\alpha^2/ds - \left(\mu w_{1\alpha^1} + w_{2\alpha^2}\right)(t)\,d\alpha^1/ds = \varphi_1(w_3)(t), \tag{3}$$

$$\mu_1\left(\psi_{1\alpha^2} + \psi_{2\alpha^1}\right)(t)\,d\alpha^2/ds - \left(\mu\psi_{1\alpha^1} + \psi_{2\alpha^2}\right)(t)\,d\alpha^1/ds = \varphi_2(t). \tag{4}$$

In (1)–(4) the following notations are used:

$$f_j \equiv f_j(w_3) = k_{j+2} w_{3\alpha^j} - w_{3\alpha^j} w_{3\alpha^j\alpha^j} - \mu_2 w_{3\alpha^{3-j}} w_{3\alpha^1\alpha^2} - \mu_1 w_{3\alpha^j} w_{3\alpha^{3-j}\alpha^{3-j}} - \beta_2 R^j,$$

$$g_j \equiv g_j(w_3) = k_0 w_{3\alpha^j} - \beta_1 L^j,\ j = 1,2,\ \mu_1 = (1-\mu)/2,\ \mu_2 = (1+\mu)/2,$$

$$\varphi_1(w_3)(t) = \beta_2 P^1(s) + \left[\mu w^2_{3\alpha^1}(t)/2 + w^2_{3\alpha^2}(t)/2\right] d\alpha^2/ds - \mu_1 w_{3\alpha^1}(t) w_{3\alpha^2}(t) d\alpha^1/ds, \tag{5}$$

$$\varphi_2(t) = \beta_1 N^2(s), t = t(s) = \alpha^1(s) + i\alpha^1(s) \in \Gamma, k_3 = k_1 + \mu k_2, k_4 = k_2 + \mu k_1,$$

$$k_5 = k_1^2 + k_2^2 + 2\mu k_1 k_2, k_0 = 6k^2(1-\mu)/h^2, \beta_1 = 12(1-\mu^2)/(h^3 E), \beta_2 = (1-\mu^2)/(Eh).$$

The system (1) together with the boundary conditions (2)–(4) describes the state of equilibrium isotropic elastic homogeneous shell with simply supported edges within the framework of Timoshenko shear model [1, pp. 168-170, 269]. Here $T^{\lambda\mu}$ are stresses $(\lambda, \mu = \overline{1,3})$; $w_j (j = 1,2)$ and w_3 are tangential and normal displacements of the points of S_0; $\psi_i (i = 1,2)$ are rotation angles of normal cross-section of S_0; $R^j\left(j = \overline{1,3}\right), L^k(k = 1,2), N^2, P^2$ are components of the external forces acting on the shell; $\mu = const$ is the Poisson coefficient, $E = const$ is Young‘s modulus, $k_1, k_2 = const$ are principal curvatures; $k^2 = const$ is the shear coefficient; $h = const$ is the shell width; α^1, α^2 are the Cartesian coordinates of the points in the domain Ω.

Problem (1)-(4). Find a solution to system (1) under boundary conditions (2)-(4).

There are a number of works devoted to the solvability of nonlinear problems in the framework of the Timoshenko displacement model [2-7]. For this purpose the theory of problems Rimann-Hilbert for holomorphic functions in the unit circle is used. Therefore, the field assumed from the beginning the unit circle [2-6], or conformal mappings on the unit circle [7]. At the present time, on the unit circle existence theorems of solutions of nonlinear problems for Timoshenko-type shell with rigidly clamped edges [2], with free edges [3] and with simply supported edges [4-6] are obtained. In [7] the system (1) is studied for shells of Timoshenko type with free edges in an arbitrary field Ω. The method of works [3], [4], [7] is developing on the case of arbitrary elastic shell with simply supported edges in this paper.

Consider boundary-value problem (1)-(4) in a generalized formulation. Let the following conditions hold true: (a) Ω is a simply connected domain with the boundary $\Gamma \in C^1_\beta$; (б) external forces $R^i\left(i = \overline{1,3}\right), L^k(k = 1,2) \in L_p(\Omega)$, $N^2, P^2 \in C_\beta(\Gamma)$; in what follows $p > 2, 0 < \beta < 1$.

Definition. The vector of generalized displacements $a = (w_1, w_2, w_3, \psi_1, \psi_2) \in W^{(2)}_p(\Omega)$, $p > 2$, is a generalized solution to the problem (1)-(4) if the vector satisfies almost everywhere the equations of system (1) and it satisfies boundary conditions (2)–(4) in pointwise fashion.

Here $W^{(2)}_p(\Omega)$ is a Sobolev space. Let us note that due to embedding theorems for Sobolev spaces $W^{(2)}_p(\Omega)$ with $p > 2$, the generalized solution α belongs to $C^1_\beta(\overline{\Omega})$. In what follos $\alpha = (p-2)/p$.

2. Solution to problem (1)-(4) with respect to tangential displacements and angles of rotation

Let us consider the first two equations in (1) and initially assume that w_3 is fixed. In terms of the complex function $\omega = w_{1\alpha^1} + w_{2\alpha^2} + i\mu_1\left(w_{2\alpha^1} - w_{1\alpha^2}\right)$ these equations can be represented in the form

$$\omega_{\bar{z}} = f, \tag{6}$$

where $f = (f_1 + f_2)/2, \omega_{\bar{z}} = \left(\omega_{\alpha^1} + i\omega_{\alpha^2}\right)/2, z = \alpha^1 + i\alpha^2$.

Equation (6) is an inhomogeneous Cauchy–Riemann equation. It has general solution [8]:

$$\omega(z) = \Phi_1(z) + Tf(z), \qquad Tf = -\frac{1}{\pi}\iint_\Omega \frac{f(\zeta)}{\zeta - z} d\xi d\eta, \zeta = \xi + i\eta, \tag{7}$$

where $\Phi_1(z)$ is an arbitrary holomorphic function that belongs to the space $C_\alpha\left(\overline{\Omega}\right)$.

It is well-known [8,pp.41,53], that Tf is a completely continuous operator which acts in $L_p(\Omega), p > 2,\ C_\alpha^k(\overline{\Omega})$. It maps these spaces into $C_\alpha(\overline{\Omega})$ and $C_\alpha^{k+1}(\overline{\Omega})$, respectively. Besides, there exist the generalized derivatives [8,pp.33-34,53-67]

$$\frac{\partial Tf}{\partial \bar{z}} = f, \frac{\partial Tf}{\partial z} \equiv Sf = -\frac{1}{\pi}\iint_\Omega \frac{f(\zeta)}{(\zeta - z)^2} d\xi d\eta, \tag{8}$$

where the integral exists in the principal value sense of Cauchy (almost everywhere when $f \in L_p(\Omega), p > 1$) and Sf is a linear bounded operator in $L_p(\Omega),\ C_\alpha^k(\overline{\Omega})$.

With the function $\omega_0(z) = w_2 + iw_1$ relation (7) can be also rewritten in the form of an inhomogeneous Cauchy–Riemann equation

$$\omega_{0\bar{z}} = i(d_1\omega + d_2\overline{\omega}) \equiv id[\omega], d_j = (\mu_1 + (-1)^j)/(4\mu_1), j = 1,2, \tag{9}$$

The general solution of this equation is

$$\omega_0(z) = \Phi_2(z) + iTd[\Phi_1 + Tf](z) \tag{10}$$

where $\Phi_2(z)$ is an arbitrary holomorphic function of the class $C_\alpha^1(\overline{\Omega})$.

Thus, for fixed w_3 the general solution of the two first equations (1) is of the form (10) and contains two arbitrary holomorphic functions $\Phi_j(z), j = 1,2.$ We define these functions so that tangential displacements w_1, w_2 will satisfy boundary conditions (2), (3). First, we find $\Phi_2(z)$ from the condition $w_1 = 0$ on Г. We have a Rimann-Hilbert problem for the holomorphic function $\Phi_2(z)$ with the boundary condition

$$\mathrm{Re}[i\Phi_2(t)] = \mathrm{Re}\, Td[\omega](t), t \in \Gamma. \tag{11}$$

Let $z = \varphi(\zeta)$ is conformal mapping of the unit disk $\overline{K} : |\zeta| \le 1$ to the area $\overline{\Omega}$. It is known that if condition (a) a function $\varphi(\zeta)$ belongs to the space $C_\beta^1(\overline{K})$, [8, p.25]. Under the conditions (11) will hold a replacement $t \to \varphi(t)$, $\Phi_2(\varphi(t)) \to \Phi_2(t)$, leaving for new variables to the previous notation. As a result, we have a Rimann-Hilbert problem for the holomorphic function $\Phi_2(z)$ of the unit disk K with the boundary condition

$$\mathrm{Re}[i\Phi_2(t)] = \mathrm{Re}\, Td[\omega](\varphi(t)), t \in \partial K : |t| = 1. \tag{12}$$

Then the solution of the Riemann-Hilbert problem (12) has the form [9, p.253]

$$\Phi_2(z) = -\frac{1}{2\pi}\int_{\partial K} \mathrm{Re}\, Td[\Phi_1 + Tf](\varphi(t))\frac{t+z}{t-z}\frac{dt}{t} + c_0, z \in \overline{K}, \tag{13}$$

where c_0 is a arbitrary real constant.

We differentiate relation (13) with respect to z, we find

$$\Phi_2'(z) = -\frac{1}{\pi\varphi'(z)}\int_{\partial K} \mathrm{Re}\, Td[\Phi_1 + Tf](\varphi(t))\frac{dt}{(t-z)^2}. \tag{14}$$

We substitute relations for the tangential displacements w_1, w_2 from (10) into (3). Hence, boundary conditions (3) take the form

$$\mathrm{Re}\{t'\Phi(t)\} = h(t), t' = dt/ds, t \in \Gamma, \tag{15}$$

where

$$h(t) = l(w_3)(t) + \mathrm{Re}\{t'Sd[\Phi_1]^+(t)\} - \mathrm{Re}\{\mu_3\bar{t}'\Phi_1(t)\}/2, \mu_3 = (1+\mu)/(2(1-\mu)), \tag{16}$$

$$l(w_3)(t) = \frac{\varphi_1(w_3)(t)}{\mu - 1} + \mathrm{Re}\{t'Sd[Tf]^+(t) - \mu_3\, d\alpha^1/ds\, \mathrm{Re}\, Tf(t)\} \equiv l[f(w_3); \varphi_1(w_3)].$$

Via the $Sd[\Phi_1]^+(t)$ means the limit of the function $Sd[\Phi_1](z)$ as $z \to t \in \Gamma$ from the interior of the domain Ω. $\Phi(t)$ the boundary value of holomorphic functions in Ω

$$\Phi(z) = i\Phi_2'(z) + \mu_3\,\Phi_1(z)/2 \tag{17}$$

Thus, for the $\Phi(z)$ we have a Rimann-Hilbert problem with the boundary condition (15) in an arbitrary field Ω. Using conformal mapping, we have a Rimann-Hilbert problem for the holomorphic function $\Phi(z)\varphi'(z)$ in the unit disk K with the boundary condition

$$\mathrm{Re}[t'\varphi'(t)\Phi(t)] = h(\varphi(t))|\varphi'(t)|, t \in \partial K : |t| = 1, \tag{18}$$

where $t' = dt / d\sigma, d\sigma$ is part of an arc of a circle ∂K.

The index of problem (18) equals -1. Therefore, the solution of this problem is [9, p.253]

$$\Phi(z) = -\frac{1}{\pi\varphi'(z)}\int_{\partial K}\frac{h(\varphi(t))|\varphi'(t)|}{t-z}\frac{dt}{t}, z \in \overline{K}, \tag{19}$$

and the solvability condition

$$\int_{\partial K}\frac{h(\varphi(t))|\varphi'(t)|}{t}dt = 0 \tag{20}$$

of problem should be fulfilled. Which can be converted to the form

$$\int_{\Gamma} P^2(s)ds + \iint_{\Omega} R^2 d\alpha^1 d\alpha^2 = 0. \tag{21}$$

We find $\Phi_1(z)$ from the relation (17). For this purpose $\Phi(z)$ and $\Phi_2'(z)$ replace it with expressions from (19) and (14). Taking into consideration $h(\varphi(t))$ in (16) and equality

$$Sd[\Phi_1]^+(\varphi(t)) = \frac{1}{2}\bar{t}^2\frac{\overline{\varphi'(t)}}{\varphi'(t)}d[\Phi_1(t)] +$$

$$+\frac{d_1}{2\pi i}\bar{t}\int_{\partial K}\frac{\overline{\varphi_0(\tau,t)}}{[\varphi_0(\tau,t)]^2}\frac{\varphi'(\tau)\bar{\tau}}{\tau-t}\Phi_1(\tau)d\tau + \frac{d_2}{2\pi i}\int_{\partial K}\frac{\overline{\varphi'(\tau)}}{\varphi_0(\tau,t)}\frac{\bar{\tau}^2}{\tau-t}\overline{\Phi_1(\tau)}d\tau, \tag{22}$$

where $\varphi_0(\tau,t) = [\varphi(\tau) - \varphi(t)]/(\tau - t)$ to $\tau \neq t$ and $\varphi_0(t,t) = \varphi'(t)$ to $\tau = t$, $d[\Phi_1(t)]$, d_j are defined in (9), using the Cauchy formula for $\Phi_1(z)$ and (4.7), (8.8a) from [8, p.28, 55], we have

$$\Phi_1(z) = \Phi_1[l(w_3)](z), z \in K, \tag{23}$$

where

$$\Phi_1[l(w_3)](z) = 2(\mu-1)S_{\partial K}(\mathrm{Re}\,Td[Tf](\varphi(\tau)))(z) + \frac{(\mu-1)}{\pi}\int_{\partial K}\frac{l(w_3)(\varphi(t))|\varphi'(t)|}{t(t-z)}dt, \tag{24}$$

$$S_{\partial K}f(z) = \frac{1}{2\pi i}\int_{\partial K}\frac{f(t)}{(t-z)^2}dt,$$

$l(w_3)$ is defined in (16).

Let $\zeta = \psi(z)$ reverse function to $z = \varphi(\zeta)$. It is known that $\psi(z) \in C_\beta^1(\overline{\Omega})$ [8, p.25]. Therefore, $\Phi_1(\psi(z)) \in W_p^{(1)}(\Omega)$, $2 < p < 2/(1-\beta)$.

We substitute expression (23) into (13) to obtain

$$\Phi_2(\psi(z)) = \Phi_2[l(w_3)](\psi(z)) + c_0, z \in \Omega, \tag{25}$$

$$\Phi_2[l(w_3)](\psi(z)) = -\frac{1}{2\pi}\int_{\partial K}(\mathrm{Re}\,Td[\Phi_1[l(w_3)]](t) + \mathrm{Re}\,Td[Tf](t))\frac{t+\psi(z)}{t-\psi(z)}\frac{dt}{t}.$$

Consider tangential displacements w_1 and w_2 that satisfy the first two equations (1) and conditions (2), (3). Upon substituting (23), (25) into (10) and assuming that condition (21) is true, we obtain

$$\omega_0(z) = H_0 w_3(z) + c_0, z \in \Omega, \tag{26}$$

$$H_0 w_3(z) \equiv H_0[f(w_3); l(w_3)](z) = \Phi_2[l(w_3)](\psi(z)) + iTd[\Phi_1[l(w_3)](\psi(\zeta)) + Tf(w_3)(\zeta)](z).$$

We now turn to functions ψ_1, ψ_2 in the last two equations (1). These functions should satisfy boundary conditions (2), (4).

Let us note that the structure of left-hand sides in the last two equations (1) coincides with the structure of left-hand sides in boundary conditions (2) and (4). Relations for tangential displacements differ only in the right-hand sides. Therefore at fixed right-hand sides for rotation angles we obtain

$$\psi = \psi_2 + i\psi_1 = H_0[g + \tilde{\psi}; l[g + \tilde{\psi}; \varphi_2]] + c_1, \tag{27}$$

$$g \equiv g(w_3) = (g_1 + ig_2)/2, \tilde{\psi} = k_0(\psi_1 + i\psi_2)/2,$$

where $g_j\,(j = 1,2)$ are defined in (5), $H_0[f; g]$ is defined in (26), $l[g + \tilde{\psi}; \varphi_2]$ is defined in (16); c_1 an arbitrary real constant.

As this takes place, the condition of solvability, similar to (20), can be reduced to the form

$$\beta_1\Big(\int_\Gamma N^2(s)ds + \iint_\Omega L^2 d\alpha^1 d\alpha^2\Big) - k_0 \iint_\Omega \psi_2 d\alpha^1 d\alpha^2 = 0, \tag{28}$$

where N^2, L^2 are components of external load.

Since $H_0[g + \tilde{\psi}; l[g + \tilde{\psi}; \varphi_2]] = H_0[g; \varphi_2] + H_0[\tilde{\psi}; 0]$, then (27) can be written in the form

$$\psi - K_0\psi = H_0[g; \varphi_2] + c_1, K_0\psi = H_0[\tilde{\psi}; 0]. \tag{29}$$

It is obvious that $K_0\psi$ are linear completely continuous operators in $W_p^{(2)}(\Omega)$, $2 < p < 2/(1-\beta)$. Let us show that the homogeneous equation $\psi - K_0\psi = 0$ has only the trivial solution in $W_p^{(2)}(\Omega)$, $2 < p < 2/(1-\beta)$. Suppose the contrary: $\psi = \psi_2 + i\psi_1 \in W_p^{(2)}(\Omega)$, $2 < p < 2/(1-\beta)$, is a nonzero solution satisfying condition (28) with $N^2 = L^2 = 0$. Obviously, the function $\psi = \psi_2 + i\psi_1$ satisfies the last two equations in system (1) with $g_1 = g_2 = 0$, the homogeneous boundary conditions $\psi_1 = 0$, and conditions (4) with $\varphi_2(t) = 0$. We multiply the last two relations in (1) by ψ_1 and ψ_2, respectively, integrate the resulting relations over the domain Ω, and add them. In view of the boundary conditions, we obtain the relation

$$\iint_\Omega \{\mu_1(\psi_{1\alpha^2} + \psi_{2\alpha^1})^2 + \mu(\psi_{1\alpha^1} + \psi_{2\alpha^2})^2 + (1-\mu)(\psi^2{}_{1\alpha^1} + \psi^2{}_{2\alpha^2}) + k_0(\psi_1^2 + \psi_2^2)\} d\alpha^1 d\alpha^2 = 0,$$

which implies that $\psi_1 = \psi_2 = 0$ in $\overline{\Omega}$.. Then there exists an inverse operator $(I-K)^{-1}$, which is bounded in $W_p^{(2)}(\Omega)$, $2 < p < 2/(1-\beta)$, and whose application to Eq. (29) gives the relation

$$\psi = (I-K)^{-1}(H_0[g; \varphi_2] + c_1). \tag{30}$$

Note that the function $\psi_* = (I-K)^{-1}c_1$ satisfies the last two equations in system (1) with $g_1 = g_2 = 0$ and the homogeneous boundary conditions, of the above. Therefore, $\psi_* = 0$ in $\overline{\Omega}$. Then from Eq. (30), we derive the unique representation via the deflection for the rotation angles,

$$\psi \equiv \psi(w_3) = (I-K)^{-1} H_0[g(w_3); \varphi_2]. \tag{31}$$

Condition (28) for the rotation angles (31) is identically satisfied. To justify this fact, it suffices to substitute the expression (31) into the last equation in system (1) and integrate the resulting relation over Ω with regard to the boundary condition (4).

3. Reduction of system (1) to a single equation for the deflection and its study

The functions w_1, w_2, ψ_1, ψ_2 included in the third equation of the system (1), we replace them with expressions from (26), (31). As a result, we obtain a nonlinear second-order partial differential equation for the deflection,

$$w_{3\alpha^1\alpha^1} + w_{3\alpha^2\alpha^2} + K_1 w_3 + G_1 w_3 = 0,$$

which is equivalent to the equation

$$w_3 + Kw_3 + Gw_3 = 0, \tag{32}$$

where

$$Kw_3 = \iint_\Omega H(\varsigma, z) K_1 w_3(\varsigma) d\xi d\eta, Gw_3 = \iint_\Omega H(\varsigma, z) G_1 w_3(\varsigma) d\xi d\eta; \tag{33}$$

$H(\varsigma, z)$ is the harmonic Green function of the Dirichlet problem for the domain Ω, $K_1 w_3$ is a linear compact operator and $G_1 w_3$ is a nonlinear bounded operator from $W_p^{(2)}(\Omega)$ into $L_p(\Omega)$, $2 < p < 2/(1-\beta)$.

Then from (33) we find that Kw_3 is a linear compact operator and Gw_3 is a nonlinear bounded operator in $W_p^{(2)}(\Omega)$, $2 < p < 2/(1-\beta)$; further , the estimate

$$\left\| G_* w_3^1 - G_* w_3^2 \right\|_{W_p^{(2)}(\Omega)} \le c\left[q_0 + (1+r)r\right] \left\| w_3^1 - w_3^2 \right\|_{W_p^{(2)}(\Omega)}, \tag{34}$$

$$q_0 = \sum_{\lambda,\mu=1}^{2} \left\| T^{\lambda\mu}(0) \right\|_{C(\overline{\Omega})} + \sum_{\lambda=1}^{2} \left\| R^\lambda \right\|_{L_p(\Omega)},$$

$$T^{\lambda\mu}(0) \equiv T^{\lambda\mu}(a(0)), \quad a(0) = (w_1(0), w_2(0), 0, \psi_1(0), \psi_2(0))$$

holds for arbitrary $w_3^j (j = 1,2) \in W_p^{(2)}(\Omega)$ that belong to the ball $\|w_3\|_{W_p^{(2)}} < r$.

Let us show that the equation

$$w_3 + Kw_3 = 0 \tag{35}$$

has only the trivial solution in $W_p^{(2)}(\Omega)$, $2 < p < 2/(1-\beta)$. Let $w_3 \in W_p^{(2)}(\Omega)$, $2 < p < 2/(1-\beta)$, is a nonzero solution of Eq. (35). By relations (26), (31), to this solution, there correspond tangential displacements $w_j(w_3)$ and rotation angles $\psi_j(w_3)$, $j = 1,2$. They satisfying the system (1) with $R_1 = R_2 = R_3 = L_1 = L_2 = 0$, in which the nonlinear terms do not exist, and the homogeneous boundary conditions (2), (3) with $\varphi_1(t) = 0$ and (4) with $\varphi_2(t) = 0$. The each of identities in (1) we multiply by $w_1, w_2, w_3, \psi_1, \psi_2$, respectively, integrate over the domain Ω, and sum the resulting relations. Then, by integrating by parts in the resulting relations, together with boundary conditions, imply that $w_3 = 0$ in $\overline{\Omega}$. Consequently, Eq. (35) has only the zero solution in $W_p^{(2)}(\Omega)$, $2 < p < 2/(1-\beta)$. Then there exists an inverse operator $(I+K)^{-1}$ bounded in $W_p^{(2)}(\Omega)$, $2 < p < 2/(1-\beta)$, which permits one to reduce Eq. (32) to the equivalent equation

$$w_3 + G_* w_3 = 0, \quad G_* w_3 = (I+K)^{-1} Gw_3, \tag{36}$$

where $G_* w_3$ is a nonlinear bounded operator in $W_p^{(2)}(\Omega)$, $2 < p < 2/(1-\beta)$; moreover, the estimate $\left\| G_* w_3^1 - G_* w_3^2 \right\|_{W_p^{(2)}(\Omega)} \le q_* \left\| w_3^1 - w_3^2 \right\|_{W_p^{(2)}(\Omega)}$, $q_* = c\left\|(I+K)^{-1}\right\|_{W_p^{(2)}(\Omega)} \left[q_0 + (1+r)r\right]$ holds for arbitrary $w_3^j (j = 1,2) \in W_p^{(2)}(\Omega)$ that lie in the ball $\|w_3\|_{W_p^{(2)}} < r$.

Suppose that the radius r of the ball and the external forces acting on the shell satisfy the conditions

$$q_* < 1, \qquad \left\| G_*(0) \right\|_{W_p^{(2)}(\Omega)} < (1 - q_*) r. \tag{37}$$

Under these conditions, for Eq. (36), one can use the contraction mapping principle [10, p. 146].

The fair following the main

Theorem. Let conditions (a), (б) in Section 1 be fulfilled and inequality (37) holds. Then condition (21) is necessary and sufficient for the solvability of the geometrically nonlinear equilibrium problem for shallow elastic isotropic homogeneous shells of the Timoshenko type under the boundary conditions (2)–(4). Then the problem has generalized solution $a = (w_1, w_2, w_3, \psi_1, \psi_2) \in W_p^{(2)}(\Omega)$, $2 < p < 2/(1-\beta)$. Components w_1, w_3, ψ_1, ψ_2 are uniquely defined and component w_2 depends on constant.

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