

One cutting plane algorithm using auxiliary functions

I Ya Zabotin and K E Kazaeva

Kazan (Volga Region) Federal University, 18, Kremlyovskaya st., Kazan, 420008,
 tel.: (843)233-71-56

E-mail: iyazabotin@mail.ru

Abstract. We propose an algorithm for solving a convex programming problem from the class of cutting methods. The algorithm is characterized by the construction of approximations using some auxiliary functions, instead of the objective function. Each auxiliary function bases on the exterior penalty function. In proposed algorithm the admissible set and the epigraph of each auxiliary function are embedded into polyhedral sets. In connection with the above, the iteration points are found by solving linear programming problems. We discuss the implementation of the algorithm and prove its convergence.

1. Introduction

In the projecting of complex technical systems solving one-criterion and multi-criteria optimization problems is often a necessity. Some examples of the optimization problem statements related to the synthesis of electronic systems and their individual subsystems can be found in [1, 2]. Often such problem statements have the form of mathematical programming problems.

Cutting plane methods form a well-known class of mathematical programming problem solving methods (e.g., [3 - 6]). Part of this class of methods uses the polyhedral approximation of the epigraph of the objective function (e.g., [3, 6 - 9]). Proposed cutting algorithm precisely refers to this group. This algorithm differs from the known ones in using approximation of the auxiliary function's epigraphs. These auxiliary functions are constructed in the form of the sum of the objective function and external penalties for the constraint region of the original problem.

2. Problem setting.

We solve the problem

$$\min \{f(x): x \in D\}, \quad (1)$$

D – a convex bounded closed set in the n -dimensional Euclidean space R_n , $f(x)$ – a convex function in R_n .

We set $f^* = \min \{f(x): x \in D\}$, $X^* = \{x \in D: f(x) = f^*\}$, $\text{epi}(g, G) = \{(x, \gamma) \in R_{n+1} : x \in G, \gamma \geq g(x)\}$, where $G \subset R_n$, $g(x)$ – the function defined in R_n . Denote by $W(z, Q) = \{a \in R_n : \langle a, u - z \rangle \leq 0 \forall u \in Q\}$ the bunch of normalized generally support vectors for the set $Q \subset R_{n+1}$ at the point $z \in R_{n+1}$, $\text{int}Q$ – interior of the set Q and $K = \{0, 1, \dots\}$.

3. The cutting algorithm and discussion

The proposed solution algorithm for the problem (1) generates a sequence of approximations $\{x_k\}$, $k \in K$, by the following rule.



Choose point

$$v \in \text{int epi}(f, D).$$

Set a convex penalty function $P_0(x)$ with the conditions $P_0(x) = 0$ for all $x \in D$ and $P_0(x) > 0$ for all $x \notin D$. Put

$$F_0(x) = f(x) + P_0(x).$$

Construct a convex bounded closed set $D_0 \subset R_n$ and convex closed set $M_0 \subset R_{n+1}$ such that

$$D \subset D_0, \text{epi}(F_0, R_n) \subset M_0.$$

Fix a number

$$\gamma \leq \min \{f(x): x \in D_0\} = f_0^*$$

and a positive numerical sequence $\{\Delta_k\}$, $k \in K$. Set $i = 0$, $k = 0$.

1. Find a solution $u_i = (y_i, \gamma_i)$, where $y_i \in R_n$, $\gamma_i \in R_1$, of the following problem

$$\min \{ \gamma: (x, \gamma) \in M_i, x \in D_0, \gamma \geq \bar{\gamma} \}. \quad (2)$$

If $u_i \in \text{epi}(f, D)$, then $y_i \in X^*$ - a solution of problem (1), and the process is over.

2. If

$$F_i(y_i) - \gamma_i > \Delta_k, \quad (3)$$

then put

$$P_{i+1}(x) = P_i(x), F_{i+1}(x) = F_i(x),$$

and go to step 4. Otherwise set a convex penalty function $P_{i+1}(x)$ such that $P_{i+1}(x) = 0$ for all $x \in D$ and $P_{i+1}(x) > P_i(x)$ for all $x \notin D$. Put

$$F_{i+1}(x) = f(x) + P_{i+1}(x),$$

and go to step 3.

3. Let $i_k = i$,

$$x_k = y_{i_k}, \sigma_k = \gamma_{i_k}, \quad (4)$$

increase the value of k by one.

4. Find a point $v_i \in R_{n+1}$ as the intersection point of the segment $[v, u_i]$ and the boundary of the set $\text{epi}(F_i, R_n)$, choose a vector $a_i \in W(v_i, \text{epi}(F_i, R_n))$.
5. Let

$$M_{i+1} = M_i \cap \{u \in R_{n+1}: \langle a_i, u - v_i \rangle \leq 0\}.$$

Increase the value of i by one and go to step 1.

Let us make some remarks concerning the algorithm.

If the sets D_0 and M_0 are polyhedral, then the problems of constructing approximations u_i (2) are linear programming problems, for all $i \in K$. If put $M_0 = R_{n+1}$ then the pair $(y_0, \bar{\gamma})$, where $y_0 \in D_0$, can be a solution of problem (2) for $i = 0$.

Let $x^* \in X^*$. As $x^* \in D_0$ and the choice of number $\bar{\gamma}$ can be made by the inequalities $f^* \geq f_0^* \geq \bar{\gamma}$, it is easy to prove that $(x^*, f^*) \in M_i$, $i \in K$, i.e. the admissible set of the problem (2) is not empty. The last inclusion implies the inequality

$$\gamma_i \leq f^*, i \in K. \quad (5)$$

The proof of the following criterion of optimality (from step 1) is based on this inequality.

Theorem 1. If the inclusion $u_i \in \text{epi}(f, D)$ occurs for some $i \in K$, then $y_i \in X^*$.

Notice that some methods of specifying penalty functions can be found, for example, in [10, 11].

We move on to the investigation of the convergence of the algorithm. Firstly note the boundness of the sequences $\{u_i\}$ and $\{v_i\}$ constructed by the algorithm, because the inclusions $y_i \in D_0$ and the inequalities (5) are implemented for the points (y_i, γ_i) .

Show that with a sequence $\{u_i\}$, $i \in K$, the algorithm constructs sequences $\{x_k\}$, $\{\sigma_k\}$, $k \in K$, too.

Lemma 1. If the sequence $\{u_i\}$, $i \in K$ is constructed by the described algorithm, there is an index $i_k \in K$ which satisfies (4) for every $k \in K$.

Proof. Fix an index $k \in K$ and show the existence of such index $i_k \in K$ for which the inequality $F_{i_k}(y_{i_k}) - \gamma_{i_k} \leq \Delta_k$ holds. Thereby the equalities (4) will be proved for the selected $k \in K$.

Assume the contrary, i.e. (3) is executed for the fixed Δ_k and for all $i \in K$. Then, according to the step 2 of the algorithm, we have the equalities $P_i(x) = P_0(x)$, $F_i(x) = F_0(x)$ for all $i > 0$. It means that

$$F_0(y_i) - \gamma_i > \Delta_k \quad \forall i \in K. \quad (6)$$

From the sequence $\{u_i\}$, $i \in K$, distinguish a convergent subsequence $\{u_i\}$, $i \in K' \subset K$, and let $u' = (y', \gamma')$ be its limit point. Then referring to inequalities (6)

$$F_0(y') - \gamma' \geq \Delta_k. \quad (7)$$

Prove that the following equality holds for this subsequence:

$$\lim_{i \in K'} \|v_i - u_i\| = 0 \quad (8)$$

By the choice of the points v_i , there is such $\mathcal{T}_i \in [0, 1)$ for each $i \in K$ that

$$v_i = u_i + \mathcal{T}_i(v - u_i). \quad (9)$$

Fix $i', i'' \in K'$ so that $i'' > i'$. By the construction $M_{i''} \subset M_{i'}$ and consequently $a_{i'} \in W(v_{i'}, M_{i''})$. In view of inclusion $u_{i''} \in M_{i''}$ we have the inequality $\langle a_{i'}, u_{i''} - v_{i'} \rangle \leq 0$. Hence from (9) there is the inequality

$$\langle a_{i'}, u_{i'} - u_{i''} \rangle \geq \mathcal{T}_{i'} \langle a_{i'}, u_{i'} - v \rangle. \quad (10)$$

As $v \in \text{int epi}(f, D)$ and $v_i \notin \text{int epi}(f, D)$, $i \in K$, by lemma 1 from [12] there is a number $\delta > 0$ such that $\langle a_i, v - v_i \rangle \leq -\delta$ for all $i \in K$. By the equality (9) and the inequality $0 \leq \mathcal{T}_i < 1$, $i \in K$, we have the inequality $\langle a_i, v - v_i \rangle \leq -\delta$, $i \in K$. Then, from (10) we obtain $\langle a_{i'}, u_{i'} - u_{i''} \rangle \geq \mathcal{T}_{i'} \delta$ or $\|u_{i'} - u_{i''}\| \geq \mathcal{T}_{i'} \delta$. Due to convergence of the sequence $\{u_i\}$, $i \in K'$, the limit relation $\mathcal{T}_i \rightarrow 0$, $i \in K'$, follows from the last inequality. By the boundness of $\{\|v - u_i\|\}$, $i \in K'$, using (9), we obtain the required equality (8).

Then, by equality (8) and by the inclusion $v_i \in \text{epi}(F_0, R_n)$, $i \in K'$, there is the inclusion $u' \in \text{epi}(F_0, R_n)$, i.e.

$$F_0(y') \leq \gamma'. \quad (11)$$

Contrariwise, $\text{epi}(F_0, R_n) \subset M_i$, $i \in K$. It means that the point $(y_i, F_0(y_i))$, $i \in K$, is a permissible solution of the problem (2) for each $i \in K$. Then the inequality $\gamma_i \leq F_0(y_i)$ holds for the solution (y_i, γ_i) of the problem (2) for each $i \in K$. Passing in this inequality to the limit $i \in K'$ we get $\gamma' \leq F_0(y')$. Adding (11) we obtain equality $F_0(y') = \gamma'$ which contradicts (7). QED

Theorem 2. Let the functions $P_i(x)$, $i \in K$, are chosen in the algorithm on condition

$$\lim_{i \in K} P_i(x) = +\infty \quad \forall x \notin D, \quad (12)$$

$\{(x_k, \sigma_k)\}$, $k \in K' \subset K$, is a convergent subsequence of the sequence $\{(x_k, \sigma_k)\}$, $k \in K$, and $\bar{u} = (\bar{x}, \bar{\sigma})$ is its limit point. Then $\bar{x} \in X^*$, $\bar{\sigma} \in f^*$.

Proof. Remind that according to (4) $(x_k, \sigma_k) = (y_{i_k}, \gamma_{i_k}) = u_{i_k}$, $k \in K$. By the method of justification of equality (8) it is proved that

$$\lim_{k \in K'} \|v_{i_k} - u_{i_k}\| = 0 \quad (13)$$

Distinguish a convergent subsequence $\{v_{i_k}\}$, $k \in K'' \subset K'$, of the sequence $\{v_{i_k}\}$, $k \in K'$. Let $\bar{v} = (\bar{w}, \bar{\alpha})$, where $\bar{w} \in R_n$ and $\bar{\alpha} \in R_1$ is its limit point. It is not difficult to check the validity of inequality $f(\bar{w}) \leq \bar{\alpha}$ for this point. Moreover, considering the terms of (12) the inclusion $w \in D$ is proved. Hence, $v \in \text{epi}(f, D)$. Then from (13) we have the inclusion $u \in \text{epi}(f, D)$, i.e. $\bar{x} \in D$ и $f(\bar{x}) \leq \bar{\sigma}$. But according to (5) $\sigma \leq f^*$. Consequently, we get the inequalities $f(\bar{x}) \leq \bar{\sigma} \leq f^* \leq f(\bar{x})$ from which the assertion of theorem follows.

Remark that the convergence theorem of the algorithm is proved without any additional requirements for the choice of the sequence $\{\Delta_k\}$, $k \in K$. In particular it is permitted to put $\Delta_k = \Delta > 0$ for all $k \in K$. If Δ is arbitrarily large, the inequality $F_i(y_i) - \gamma_i < \Delta$ holds for all $i \in K$. In this case the functions $P_{i+1}(x)$ differ from $P_i(x)$ and $x_k = y_i$, $k \in K$.

If we set the sequence $\{\Delta_k\}$ with the condition $\Delta_k \rightarrow 0$, $k \rightarrow \infty$, algorithm can be considered as the implementation of the penalty function method, where iteration points are found on the set of auxiliary functions D_0 from the condition of the approximate minimum ([10]. c. 380).

References

- [1] Zabotin I Ya and Kobchikov A V 1990 On the Choice of the Optimal Ratio between the Masses of the Electronic System Blocks (in Russian) *Kazan university* vol **18** pp 64 – 70
 - [2] Zabotin I Ya and Kobchikov A V 1991 On the Realization of Some Methods Applied to Design Optimization Problems *Avtomat. i Telemekh.* no **1** pp 169 – 172
 - [3] Bulatov V P 1977 *Embedding methods in optimization problems* (in Russian) (Novosibirsk Nauka) 161 p.
 - [4] Zabotin I Ya and Yarullun R S 2013 One approach to constructing cutting algorithms with dropping of cutting planes *Russian Math Iz. VUZ Allerton Press Inc.* vol **57** no 3 pp 60–64
 - [5] Zabotin I Ya and Yarullun R S 2013 A Cutting Method for Finding Discrete Minimax with Dropping of Cutting Planes *Lobachevskii Journal of Mathematics* vol **35** no 2 pp 157 - 163
 - [6] Nesterov Yu E 2010 *Introduction to convex optimization* (in Russian). (Moscow: MCCME) p 274
 - [7] Zabotin I Ya, Shulgina O N and Yarullun R S 2014 A Cutting Method and Construction of Mixed Minimization Algorithms on Its Basis (in Russian) *Uchenye Zapiski Kazanskogo Universiteta Ser. Fiz.-Matem. Nauki* vol **156** no 4 pp 14 – 24.
 - [8] Zabotin I Ya and Yarullun R S 2015 A Cutting-Plane Method Based on Epigraph Approximation with Discarding the Cutting Planes *Automation and Remote Control* vol **76** no 11 pp 1966–1975
 - [9] Polyak B T 1983 *Vvedenie v optimizatsiyu (Introduction to Optimization)* (Moscow: Nauka) p 384
 - [10] Vasil'ev F P 2011 *Optimization methods* (in Russian) (Moscow: MCCME) p 620
 - [11] Konnov I V 2013 *Nonlinear optimization and variational inequalities* (in Russian) (Kazan: Kazan university) p 508
- Zabotin I Ya 2011 Some embedding-cutting algorithms for mathematical programming problems (in Russian) *Izv. Irkutsk. Gos. Univ. Ser. Matem.* vol **4** no 2 pp 91–101