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Existence of solutions for electron balance problem in the stationary radio-frequency induction discharges

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Abstract. A sufficient condition for the existence of a minimal eigenvalue corresponding to a positive eigenfunction of an eigenvalue problem with nonlinear dependence on the parameter for a second order ordinary differential equation is established. The initial problem is approximated by the finite element method. Error estimates for the approximate minimal eigenvalue and corresponding positive eigenfunction are derived. Problems of this form arise in modelling the plasma of a radio-frequency discharge at reduced pressure.

1. Statement of the problem

In the present paper we investigate the problem of finding the minimal eigenvalue $\lambda \in \Lambda$, $\Lambda = [0, \infty)$, corresponding to a positive eigenfunction $u(x)$, $x \in \Omega$, $\Omega = (0, \pi)$, $\bar{\Omega} = [0, \pi]$, of the following eigenvalue problem

$$\begin{aligned} &-(p(\lambda s(x))u')' = r(\lambda s(x))u, \quad x \in \Omega, \\ &u(0) = u(\pi) = 0. \end{aligned} \tag{1}$$

Assume that $p(\mu)$, $r(\mu)$, $\mu \in \Lambda$, and $s(x)$, $x \in \bar{\Omega}$, are continuous positive functions, $p'(\mu)$, $r'(\mu)$, $\mu \in \Lambda$, are continuous functions, $p(\mu)$, $\mu \in \Lambda$, is nondecreasing bounded, $r(\mu)$, $\mu \in \Lambda$, is nondecreasing unbounded.

For fixed $\mu \in \Lambda$, by $\gamma(\mu)$ we denote the minimal simple eigenvalue corresponding to a positive eigenfunction $u(x) = u_\mu(x)$, $x \in \Omega$, of the parametric linear eigenvalue problem

$$\begin{aligned} &-(p(\mu s(x))u')' = \gamma(\mu) r(\mu s(x))u, \quad x \in \Omega, \\ &u(0) = u(\pi) = 0. \end{aligned} \tag{2}$$

Then the minimal eigenvalue λ of problem (1) is the minimal root of the following equation

$$\gamma(\mu) = 1, \quad \mu \in \Lambda. \tag{3}$$

In Section 2, assuming the condition $p(\xi s_1) > r(\xi s_2)$ for some $\xi \in \Lambda$, where s_1 and s_2 are the minimum and maximum of the function $s(x)$, $x \in \bar{\Omega}$, we prove the existence of a minimal simple eigenvalue of problem (1) being a minimal root of equation (3). In Section 3, we define a mesh scheme of the finite element method for solving problem (1). We prove the existence of a minimal simple approximate eigenvalue corresponding to a positive eigenfunction under the condition $p(\xi s_1) > r(\xi s_2)$ for some $\xi \in \Lambda$ and establish error estimates for the approximate minimal eigenvalue and corresponding positive eigenfunction. These results develop the results of the paper [1].

Problems of the form (1) arise in modelling the plasma of a radio-frequency discharge at reduced pressure. The sufficient condition obtained in the paper defines a condition necessary for maintaining a stationary inductive coupled radio-frequency discharge at reduced pressure [1–5].

Eigenvalue problems with nonlinear dependence on the parameter arise in various fields of science and technology [6–18]. Numerical algorithms for solving matrix nonlinear eigenvalue problems were constructed and investigated in [10,19–23]. Mesh methods for solving differential eigenvalue problems with nonlinear dependence on the spectral parameter were studied in [24–28]. The theoretical basis for the study of eigenvalue problems with nonlinear dependence on the parameter is results obtained for linear eigenvalue problems [29–35].

2. Existence of solutions

Let $H = L_2(\Omega)$ denotes the real Lebesgue space with norm $|\,.\,|_0$. By $V = \{v:\ v,\ v' \in H,\ v(0) = v(\pi) = 0\}$ we denote the real Sobolev space with norm $|\,.\,|_1$. Here we use the notation from [1]. For fixed $\mu \in \Lambda,$ we introduce the bilinear forms

$$a(\mu,u,v) = \int_0^\pi p(\mu s(x))u'v'\mathrm{d}x, \quad b(\mu,u,v) = \int_0^\pi r(\mu s(x))uv\mathrm{d}x,$$

for $u,v \in V$ and the Rayleigh functional $R(\mu,v) = a(\mu,v,v)/b(\mu,v,v)$ for any $v \in V\setminus\{0\}$. Put $K = \{v:\ v \in V,\ v(x) > 0,\ x \in \Omega\}.$

The differential eigenvalue problem (1) is equivalent to the variational nonlinear eigenvalue problem: find minimal $\lambda \in \Lambda$ and $u \in K,\ b(\lambda,u,u) = 1,$ such that

$$a(\lambda,u,v) = b(\lambda,u,v) \quad \forall v \in V. \tag{4}$$

For fixed $\mu \in \Lambda,$ the differential eigenvalue problem (2) is equivalent to the variational linear eigenvalue problem: find minimal $\gamma(\mu)$ and $u = u_\mu \in K,\ b(\mu,u,u) = 1,$ such that

$$a(\mu,u,v) = \gamma(\mu)b(\mu,u,v) \quad \forall v \in V. \tag{5}$$

Theorem 1. Suppose that $p(\xi s_1) > r(\xi s_2)$ for some $\xi \in \Lambda,$ where s_1 and s_2 are the minimum and maximum of the function $s(x),\ x \in \bar{\Omega}.$ Then there exists a minimal simple eigenvalue of problem (4) corresponding to a positive eigenfunction.

Proof. According to the variational characterization for the minimal simple eigenvalue of problem (5), we derive

$$\gamma(\xi) = \min_{v\in V\setminus\{0\}} R(\xi,v) = \min_{v\in V\setminus\{0\}} \frac{\int_0^\pi p(\xi s(x))(v')^2\mathrm{d}x}{\int_0^\pi r(\xi s(x))v^2\mathrm{d}x} \geq \frac{p(\xi s_1)}{r(\xi s_2)} \min_{v\in V\setminus\{0\}} \frac{\int_0^\pi (v')^2\mathrm{d}x}{\int_0^\pi v^2\mathrm{d}x} = \frac{p(\xi s_1)}{r(\xi s_2)} > 1.$$

By [1], $\gamma(\mu)$, $\mu\in\Lambda$, is the continuous function, and $\gamma(\mu)\to 0$ as $\mu\to\infty$. This implies the existence of a minimal root of equation (3), which defines the minimal eigenvalue λ of problem (4) corresponding to a positive eigenfunction. Thus, the theorem is proved.

3. Approximation of solutions

Define the partition of the interval $[0,\pi]$ by equidistant points $x_i = ih$, $i=0,1,\dots,N$, into the elements $e_i=(x_{i-1},x_i)$, $i=1,2,\dots,N$, $h=\pi/N$. By V_h denote the subspace of the space V consisting of continuous functions v^h linear on each element e_i, $i=1,2,\dots,N$. Set $K_h=\{v^h:\ v^h\in V_h,\ v^h(x)>0,\ x\in\Omega\}$.

The variational nonlinear eigenvalue problem (4) is approximated by the following finite-dimensional nonlinear eigenvalue problem: find minimal $\lambda^h\in\Lambda$ and $u^h\in K_h$, $b(\lambda^h,u^h,u^h)=1$, such that

$$a(\lambda^h,u^h,v^h)=b(\lambda^h,u^h,v^h)\quad \forall v^h\in V_h. \tag{6}$$

For fixed $\mu\in\Lambda$, the variational linear eigenvalue problem (5) is approximated by the following finite-dimensional linear eigenvalue problem: find minimal $\gamma^h(\mu)$ and $u^h=u^h_\mu\in K_h$, $b(\mu,u^h,u^h)=1$, such that

$$a(\mu,u^h,v^h)=\gamma^h(\mu)b(\mu,u^h,v^h)\quad \forall v^h\in V_h. \tag{7}$$

Then the minimal eigenvalue λ^h of problem (6) is the minimal root of the following equation

$$\gamma^h(\mu)=1,\quad \mu\in\Lambda. \tag{8}$$

Theorem 2. Suppose that $p(\xi s_1)>r(\xi s_2)$ for some $\xi\in\Lambda$, where s_1 and s_2 are the minimum and maximum of the function $s(x)$, $x\in\bar{\Omega}$. Then there exists a minimal simple eigenvalue of problem (6) corresponding to a positive eigenfunction.

Proof. Using variational characterization for the minimal simple eigenvalue of problem (7), we get

$$\gamma^h(\xi)=\min_{v^h\in V_h\setminus\{0\}}R(\xi,v^h)=\min_{v^h\in V_h\setminus\{0\}}\frac{\int_0^\pi p(\xi s(x))((v^h)')^2\mathrm{d}x}{\int_0^\pi r(\xi s(x))(v^h)^2\mathrm{d}x}\ge\frac{p(\xi s_1)}{r(\xi s_2)}\min_{v\in V\setminus\{0\}}\frac{\int_0^\pi (v')^2\mathrm{d}x}{\int_0^\pi v^2\mathrm{d}x}=\frac{p(\xi s_1)}{r(\xi s_2)}>1.$$

According to [1], $\gamma^h(\mu)$, $\mu\in\Lambda$, is the continuous function, and $\gamma^h(\mu)\to 0$ as $\mu\to\infty$. This implies the existence of a minimal root of equation (8), which defines the minimal eigenvalue λ^h of problem (6) corresponding to a positive eigenfunction. Thus, the theorem is proved.

By c we denote various positive constants independent of h. For fixed $\mu\in\Lambda$, we introduce an operator $P_h(\mu)\colon V\to V_h$ by the rule $a(\mu,u-P_h(\mu)u,v^h)=0$ for any $v^h\in V_h$, where $u\in V$, $|u-P_h(\mu)u|_0\le ch^2$. Put $P_h=P_h(\lambda)$, $\|v\|^2_{b(\mu)}=b(\mu,v,v)$, $v\in V$, $\mu\in\Lambda$.

By $\gamma_i^h(\mu)$, $u_i^h(\mu)=u_i^h(\mu,x)$, $x\in\bar{\Omega}$, $\mu\in\Lambda$, $i=1,2,\dots,N$, we denote eigenvalues and eigenfunctions satisfying (7) and such that

$$\gamma_1^h(\mu)<\gamma_2^h(\mu)<\dots<\gamma_N^h(\mu),$$

$$a(\mu,u_i^h(\mu),u_j^h(\mu))=\gamma_i^h(\mu)\delta_{ij},\quad b(\mu,u_i^h(\mu),u_j^h(\mu))=\delta_{ij},\quad i,j=1,2,...,N,$$

$u_1^h(\mu)=u_\mu^h\in K_h,$ $\gamma_1^h(\mu)=\gamma^h(\mu),$ the functions $u_i^h(\mu),$ $\mu\in\Lambda,$ $i=1,2,...,N,$ form a complete system in the space $V_h.$ For fixed $\mu\in\Lambda$ and sufficiently small $h,$ the estimates hold:

$$0\le\gamma_i^h(\mu)-\gamma_i(\mu)\le ch^2,\quad i=1,2,\quad |u_\mu^h-u_\mu|_0\le ch^2.$$

Theorem 3. Let λ be the minimal simple eigenvalue of problem (4) corresponding to the positive eigenfunction $u,$ and let λ^h be the minimal simple eigenvalue of problem (6) corresponding to the positive eigenfunction $u^h.$ Assume that $\gamma'(\lambda)\ne 0.$ Then the following error estimates hold $0\le\lambda^h-\lambda\le ch^2,$ $|u^h-u|_0\le ch^2,$ for sufficiently small $h.$

Proof. First estimate follows from relations

$$c_1(\lambda^h-\lambda)\le-(\gamma^h(\xi^h))'(\lambda^h-\lambda)=\gamma^h(\lambda)-\gamma^h(\lambda^h)=\gamma^h(\lambda)-\gamma(\lambda)\le c_2h^2$$

for some ξ^h and sufficiently small $h.$

Let us prove second estimate. Set $\beta_i^h=b(\lambda^h,P_hu,y_i^h),$ $i=1,2,...,N,$ where $y_i^h=u_i^h(\lambda^h),$ $i=1,2,...,N.$ Since elements $y_i^h,$ $i=1,2,...,N,$ form an orthonormal basis in the space $V_h,$ it follows that the element $P_hu\in V_h$ can be represented in the form $P_hu=\beta_1^hy_1^h+w_1^h,$ where $w_1^h=\beta_2^hy_2^h+...+\beta_N^hy_N^h.$ The inequality $\gamma_2(\lambda)-\gamma_1(\lambda)>0$ implies that $\gamma_2^h(\lambda^h)-1\ge c$ for sufficiently small $h.$ Denote

$$\zeta_h(u)=\sup_{v^h\in V_h\setminus\{0\}}\frac{|a(\lambda^h,P_hu,v^h)-\lambda b(\lambda^h,P_hu,v^h)|}{|v^h|_1}.$$

Then $\zeta_h(u)\le ch^2.$

To show the estimate $|w_1^h|_1\le ch^2$ for sufficiently small $h,$ we note that

$$a(\lambda^h,P_hu,w_1^h)=a(\lambda^h,w_1^h,w_1^h),$$

$$b(\lambda^h,P_hu,w_1^h)=b(\lambda^h,w_1^h,w_1^h),$$

$$a(\lambda^h,w_1^h,w_1^h)\ge\gamma_2^h(\lambda^h)b(\lambda^h,w_1^h,w_1^h).$$

Hence we get the relations

$$|w_1^h|_1\,\zeta_h(u)\ge a(\lambda^h,P_hu,w_1^h)-b(\lambda^h,P_hu,w_1^h)=a(\lambda^h,w_1^h,w_1^h)-b(\lambda^h,w_1^h,w_1^h)\ge$$

$$\ge\frac{\gamma_2^h(\lambda^h)-1}{\gamma_2^h(\lambda^h)}a(\lambda^h,w_1^h,w_1^h)\ge c^{-1}|w_1^h|_1^2,$$

which imply the desired estimate: $|w_1^h|_1\le c\zeta_h(u)\le ch^2.$ Therefore $|P_hu-\beta_1^hy_1^h|_0\le ch^2.$

Moreover, we have

$$\beta_1^h=\|\beta_1^hy_1^h\|_{b(\lambda^h)}\le\|u\|_{b(\lambda)}+\Big|\|u\|_{b(\lambda)}-\|u\|_{b(\lambda^h)}\Big|+\|u-\beta_1^hy_1^h\|_{b(\lambda^h)}\le 1+ch^2,$$

$$\beta_1^h = \| \beta_1^h y_1^h \|_{b(\lambda^h)} \geq \| u \|_{b(\lambda)} - \left| \| u \|_{b(\lambda)} - \| u \|_{b(\lambda^h)} \right| - \| u - \beta_1^h y_1^h \|_{b(\lambda^h)} \geq 1 - ch^2,$$

since

$$\left| \| u \|_{b(\lambda)} - \| u \|_{b(\lambda^h)} \right| \leq \frac{\| u \|_{b(\lambda)}^2 - \| u \|_{b(\lambda^h)}^2}{\| u \|_{b(\lambda)} + \| u \|_{b(\lambda^h)}} \leq c(\lambda^h - \lambda) \leq ch^2,$$

$$\| u - \beta_1^h y_1^h \|_{b(\lambda^h)} \leq \sqrt{\beta_2}\, | u - \beta_1^h y_1^h |_0 \leq \sqrt{\beta_2} (| u - P_h u |_0 + | P_h u - \beta_1^h y_1^h |_0) \leq ch^2,$$

for $\beta_2 = r(\lambda s_1 + 1)$ and sufficiently small h. Consequently, we derive $|1 - \beta_1^h| \leq ch^2$.

As a result, we conclude

$$\sqrt{\beta_1}\, | u^h - u |_0 \leq \| u - y_1^h \|_{b(\lambda^h)} \leq \| u - \beta_1^h y_1^h \|_{b(\lambda^h)} + \| y_1^h - \beta_1^h y_1^h \|_{b(\lambda^h)} =$$

$$= \| u - \beta_1^h y_1^h \|_{b(\lambda^h)} + |1 - \beta_1^h| \leq ch^2,$$

where $\beta_1 = r(0)$. This completes the proof of the theorem.

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