

Eigenvectors method in mechanics of solids one-dimensional linear boundary problems solution

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Abstract. Problems of deformed elements stress-strain state analysis for small displacements, angles of rotation and deformations are considered. Eigenvectors method is suggested for solution of corresponding linear boundary problems. The method is based on lattice functions theory and differential equations theory, allowing to obtain analytical form of integral matrices. One of the modifications of the suggested method allows obtaining solution with the precision equal to computer zero. An example illustrating advantages of eigenvectors method is represented.

1. Problem statement

For all known linear models of deformed elements the system of equations describing their deformed state includes three types of linear relations: differential equilibrium equations, differential and algebraic geometry relations and physical relations [1]. For stationary one-dimensional problems these equations can reduce to the system of linear ordinary differential equations in some vector $\mathbf{y}(x, \mu)$ of dimension m :

$$\mathbf{y}'(x, \mu) = \mathbf{A}(x, \mu)\mathbf{y} + \mathbf{b}(x, \mu) \quad (1)$$

Here prime mark denotes derivative with respect to x , $\mathbf{A}$ is square matrix of dimension m , vector $\mathbf{b}$ describes the loading of the deformed element, μ is the vector of problem parameters. The form of the matrix and vectors depends on the type of the considered deformed element.

As a rule, boundary problem for the system (1) is solved on the interval $[x_1, x_2]$ with boundary conditions

$$\mathbf{C}_i \mathbf{y}_i = \mathbf{d}_i, \quad (2)$$

Here $\mathbf{y}_i = \mathbf{y}(x_i)$, $i = 1, 2$, $\mathbf{C}_i$ are matrices of dimensions $m \times m_i$ ($m_1 + m_2 = m$), $\mathbf{d}_i$ are vectors of inhomogeneous components of boundary conditions.

2. Initial parameters method of boundary problem solution

For solution of the problem (1)-(2) we shall use initial parameters method [2]. According to this method, we represent the unknown vector using its value on the left boundary of the integration interval as follows:



$$\mathbf{y}(x) = \mathbf{M}(x)\mathbf{y}_1 + \mathbf{H}(x) \quad (3)$$

In this representation matrix $\mathbf{M}(x)$ and vector $\mathbf{H}(x)$ are solutions of the following Cauchy problems:

$$\mathbf{M}'(x) = \mathbf{A}(x)\mathbf{M}(x), \quad \mathbf{M}(x_1) = \mathbf{E} \quad (4)$$

$$\mathbf{H}'(x) = \mathbf{A}(x)\mathbf{H}(x) + \mathbf{b}(x), \quad \mathbf{H}(x_1) = \mathbf{0} \quad (5)$$

Vector $\mathbf{y}(x)$ represented in such a way satisfies the system of equations (1).

Substituting vector (3) in boundary conditions (2) we obtain the system of linear algebraic equations in vector $\mathbf{y}_1$:

$$\begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2\mathbf{M}(x_2) \end{bmatrix} \mathbf{y}_1 = \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 - \mathbf{C}_2\mathbf{H}(x_2) \end{bmatrix} \quad (6)$$

So boundary problem (1)-(2) reduces to $(m+1)$ Cauchy problems in m columns of matrix $\mathbf{M}(x)$ and vector $\mathbf{H}(x)$. We shall write these problems in the following common form:

$$\mathbf{y}'(x) = \mathbf{A}(x)\mathbf{y}(x) + \mathbf{b}(x), \quad \mathbf{y}(x_1) = \mathbf{y}_0 \quad (7)$$

3. Eigenvectors method of Cauchy problem solution

When solving the problem (7) we use lattice functions for all the variables. By definition [3] lattice function exists only at discrete equidistant values of independent variable x and is equal to zero between these values of the argument. Then on each period of discreteness lattice analogue of vector function representing the right hand side of the equation (7) $\mathbf{f}(x, \mathbf{y}) = \mathbf{A}(x)\mathbf{y}(x) + \mathbf{b}(x)$ can be represented by its envelope $\boldsymbol{\varphi}(\tau, \boldsymbol{\xi}) = \text{const}$, where $\boldsymbol{\xi}$ is an envelope of vector function $\mathbf{y}$ on the considered period, τ is an argument of lattice function, $\tau = x_0 + (n-1)X_0$, X_0 is the period of discreteness, n is a number of the period of discreteness.

In order to construct exponential form of the problem (7) solution based on lattice functions using we transform the right hand side of differential equation (7) as follows:

$$\mathbf{y}'(x) = \mathbf{P}(x)\mathbf{y}(x) + \boldsymbol{\varphi}^*(\tau, \boldsymbol{\xi}) \quad (8)$$

Here $\mathbf{P}(x)$ is regularizing matrix of the system (7), and $\boldsymbol{\varphi}^*(\tau, \boldsymbol{\xi}) = \boldsymbol{\varphi}(\tau, \boldsymbol{\xi}) - \mathbf{P}(\tau)\boldsymbol{\xi}$.

We seek the general solution of differential equations system (8) using method of initial parameters relation (3). Now we obtain the following homogeneous and inhomogeneous equations for matrix $\mathbf{M}(x)$ and vector $\mathbf{H}(x)$:

$$\mathbf{M}'(x) = \mathbf{P}(x)\mathbf{M}(x), \quad \mathbf{M}(x_1) = \mathbf{E} \quad (9)$$

$$\mathbf{H}'(x) = \mathbf{P}(x)\mathbf{H}(x) + \boldsymbol{\varphi}^*(\tau, \boldsymbol{\xi}), \quad \mathbf{H}(x_1) = \mathbf{0} \quad (10)$$

Equation (9) has the following form of solution [4]:

$$\mathbf{M}(x) = \boldsymbol{\Omega}(x)\boldsymbol{\Omega}^T(0) \quad (11)$$

Here $\boldsymbol{\Omega}(x) = [\boldsymbol{\gamma}_1 \exp(\lambda_1 x) \quad \dots \quad \boldsymbol{\gamma}_m \exp(\lambda_m x)]$, where $\boldsymbol{\gamma}_i$ are eigenvectors of the matrix $\mathbf{P}(x)$, λ_i are its eigenvalues.

Solution of the equation (10) for the vector $\mathbf{H}(x)$ has the form

$$\mathbf{H}(x) = \mathbf{M}(x) \int_{x_1}^x \mathbf{M}^{-1}(\zeta) \boldsymbol{\varphi}^*(\tau, \xi) d\zeta \quad (12)$$

So the general solution of the equation (7) can be written as follows:

$$\mathbf{y}(x) = \mathbf{M}(x) \left[\mathbf{y}_1 + \int_{x_1}^x \mathbf{M}^{-1}(\zeta) \boldsymbol{\varphi}^*(\tau, \xi) d\zeta \right] \quad (13)$$

Here $x \in [x_1 + iX_0, x_1 + (i+1)X_0]$, $i \in [0, N-1]$.

On the next lattice functions periods of discreteness we use fitting method [2]. On the i^{th} period of problem (7) solution parameters $\mathbf{y}_1$ and $\boldsymbol{\varphi}^*(\tau, \xi)$ in the right side of equation (13) have the values calculated by the values $\mathbf{y}(x_1 + (i-1)X_0)$ of (13) on the previous period.

There can be two variants of the suggested method. If regularizing matrix $\mathbf{P}$ is chosen constant, we calculate matrix $\mathbf{M}$ only once on the first period of discreteness and set it constant for all the solution interval. We shall call this variant global eigenvectors method. If regularizing matrix $\mathbf{P}(x)$ depends on the argument x (as a rule, Jacobian of equations system is taken as a regularizing matrix in this case), the matrix $\mathbf{M}(x)$ is calculated on each period of discreteness. We shall call this variant local eigenvectors method.

The equation (13) is an exact analytical solution of Cauchy problem (7) on any period of discreteness. Analysis of accuracy and stability of the suggested method is a separate fundamental investigation, so in this work we shall represent testing the method on a concrete example to reveal its characteristics.

4. Example

The method was tested on a variety of mathematical physics and mechanics of deformed solid linear problems. Thorough testing didn't reveal any disadvantages of the method. In order to prove correctness of the results obtained we shall consider here the problem which has an exact analytical solution in the class of continuous functions.

We consider the following system of equations:

$$\mathbf{y}' = \mathbf{A}\mathbf{y} \quad (14)$$

$$\text{Here } \mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4\beta^4 & 0 & 0 & 0 \end{pmatrix}.$$

Such a system with different meanings of vector $\mathbf{y}$ and parameter β is used in analysis of beams on elastic foundation, cylindrical and spherical shells. Krylov functions $K_0(x)$, $K_1(x)$, $K_2(x)$, $K_3(x)$ satisfy system (14) [1]. Analytical solution of the system (14) can be represented as follows:

$$\mathbf{y} = \mathbf{M}(x)\mathbf{C} \quad (15)$$

Here $\mathbf{C}$ is a vector of constants, $\mathbf{M}(x)$ is normalized integral matrix:

$$\mathbf{M}(x) = \begin{pmatrix} K_0(x) & \beta^{-1}K_1(x) & \beta^{-2}K_2(x) & \beta^{-3}K_3(x) \\ -4\beta K_3(x) & K_0(x) & \beta^{-1}K_1(x) & \beta^{-2}K_2(x) \\ -4\beta^2 K_2(x) & -4\beta K_3(x) & K_0(x) & \beta^{-1}K_1(x) \\ -4\beta^4 K_1(x) & -4\beta^2 K_2(x) & -4\beta K_3(x) & K_0(x) \end{pmatrix} \quad (16)$$

Method precision was tested for all the components of the matrix $\mathbf{M}(x)$. For comparison we used a number of well-known numerical methods, in particular, Runge-Kutta, Kutta-Merson with automatic step change, Bulirsch-Stoer for non-stiff systems of differential equations, and Radau, Bulirsch-Stoer and Rosenbrock for stiff systems of differential equations. When carrying out calculations with constant step we took 100 steps. For the methods with changing step we took relative tolerance equal to 10^{-3} . When estimating tolerances of normalized integral matrix calculation we used maximum matrix norms. Norms of relative tolerances of investigated methods are represented in table 1. Here matrix $\mathbf{M}(x)$ was calculated at $x = 0.5$ with the value of the parameter $\beta = 20$. The norms of relative tolerances for the methods used vary from $6.42 \cdot 10^{-3}$ for Kutta-Merson method to $4.58 \cdot 10^{-15}$ for the suggested in the paper local eigenvectors method.

Table 1. Relative tolerances of normalized integral matrix calculation.

Method	Norm of relative tolerance
Runge-Kutta	$3.83 \cdot 10^{-5}$
Kutta-Merson	$6.42 \cdot 10^{-3}$
Bulirsch-Stoer	$9.07 \cdot 10^{-7}$
Radau	$1.03 \cdot 10^{-5}$
Bulirsch-Stoer for stiff equations	$3.0 \cdot 10^{-4}$
Rosenbrock	$2.27 \cdot 10^{-5}$
Global eigenvectors	$1.42 \cdot 10^{-3}$
Local eigenvectors	$4.58 \cdot 10^{-15}$

It should be pointed out that methods of stiff differential equations systems solution (Radau, Bulirsch-Stoer, Rosenbrock) didn't turn out to be the best, which proves the point of view given in [5], stressing the importance of numerical experiments and their equal status with theoretical analysis of Cauchy problem solution methods characteristics.

Among the methods suggested in the paper global eigenvectors method showed accuracy slightly better than Kutta-Merson method, while local eigenvectors method allowed obtaining the solution with the accuracy equal to computer zero.

Figure 1 represents comparison of several methods (Runge-Kutta, Rosenbrock, global and local eigenvectors) relative tolerances common logarithms of calculating the function $K_0(x) = ch(x) \cdot \cos(x)$ as a component of system (14) solution. We can see that global eigenvectors method is the least precise, Rosenbrock method (as any other mentioned in the paper method of stiff differential equations solution) is insignificantly better than Runge-Kutta method, while local eigenvectors method shows essentially different order of relative tolerance, which varies from $\sim 10^{-20}$ to $\sim 10^{-15}$.

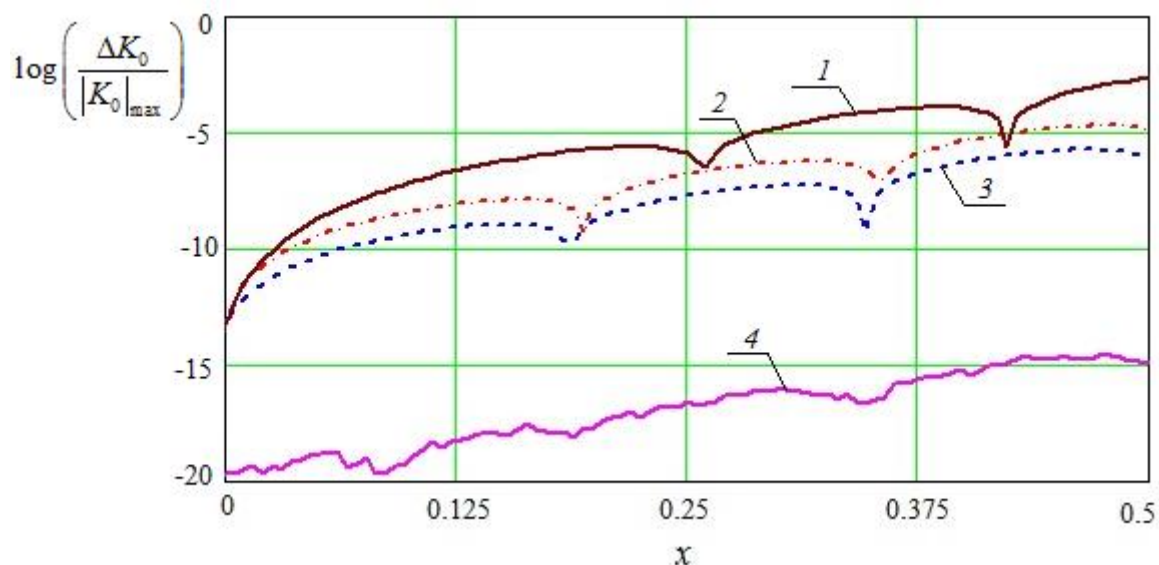


Figure 1. Comparison of relative tolerances common logarithms of function $K_0(x)$ calculation (1 – global eigenvectors method, 2 – Runge-Kutta method, 3 – Rosenbrock method, 4 – local eigenvectors method).

5. Conclusion

Eigenvectors method of deformed elements state calculation is represented. The method doesn't require any limitations for right hand side of differential equations system within the meaning [1, 2]. It allows obtaining analytical expression of the solution based on eigenvectors and eigenvalues of the equations system regularizing matrix. Local form of eigenvectors method gives the precision equal to computer zero and can be used as a basic method for solving the problems which don't have exact analytical solution.

References

- [1] Abramyan B L et al. 1968 *Strength, Stability, Vibrations* (Moscow: Mashinostroenie) [in Russian]
- [2] Biderman V L 1977 *Mechanics of Thin Structures* (Moscow: Mashinostroenie) [in Russian]
- [3] Arens V D, Fedorov S M, Khitrik M S and Luchko S V 1972 *Dynamics of Rockets with Board Digital Computers Guidance Systems* (Moscow: Mashinostroenie) [in Russian]
- [4] Strazheva I V and Melkumov V S 1973 *Vector-matrix Methods in Flight Mechanics* (Moscow: Mashinostroenie) [in Russian]
- [5] Hall J and Watt J M 1976 *Modern Numerical Methods for Ordinary Differential Equations* (Oxford: Clarendon Press)