

Calculation of elastic-plastic deformations by FEM

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Abstract. The article is devoted to elasto-plastic analysis for finite deformations, large displacements and rotations. An incremental method is used. The stressed state is represented by Cauchy stress and objective Jaumann rate of Cauchy stress. The von Mises yield criterion and radial return method are applied.

1. Introduction

There are many publications, where solutions of nonlinear problems of a solid mechanic are discussed, for example [1–17]. In this paper the algorithm of numerical solution of the problem of large elastic-plastic deformations is considered [1–8, 13–15]. Kinematics of a medium is described by the deformation rate tensor, the stress state is determined by the Cauchy stress tensor. The theory of flow is used for describing plastic deformation. The total deformation rate is represented as a sum of elastic and plastic parts. The solution algorithm is based on an Update Lagrange formulation. The principle of virtual work in terms of the virtual velocity is used. The numerical implementation is based on the method of finite elements (FEM) is used.

2. Kinematics

We assume that at the current time process of deformation represents the continua with elastoplastic properties. This corresponds to the Euler approach, widely used in continuum mechanics. In line with this, the kinematics of deformation of the body will be determined using the following tensors [22–26]: velocity gradient tensor $\mathbf{h} = h_{ij} \mathbf{e}_i \mathbf{e}_j$, deformation rate tensor $\mathbf{d} = \frac{1}{2} [\mathbf{h} + \mathbf{h}^T] = d_{ij} \mathbf{e}_i \mathbf{e}_j$, rate of rotation tensor $\boldsymbol{\omega} = \frac{1}{2} [\mathbf{h} - \mathbf{h}^T] = \omega_{ij} \mathbf{e}_i \mathbf{e}_j$.

3. Stress–strain relationship

The state of a stress in a body is defined by Cauchy stress tensor $\boldsymbol{\Sigma} = \sigma_{ij} \mathbf{e}_i \mathbf{e}_j$. Also we use objective Jaumann rate of Cauchy stress $\boldsymbol{\Sigma}^J$ [22–24]

$$\boldsymbol{\Sigma}^J = \dot{\boldsymbol{\Sigma}} - \boldsymbol{\omega} \cdot \boldsymbol{\Sigma} + \boldsymbol{\Sigma} \cdot \boldsymbol{\omega}, \quad (1)$$

where $\dot{\boldsymbol{\Sigma}}$ is the rate of Cauchy stress.



Consider isotropic material and denote $d_0 = \frac{1}{3}d_{ii}$, $\sigma_0^J = \frac{1}{3}\sigma_{ii}^J$, $d'_{ij} = d_{ij} - \delta_{ij}d_0$, $\sigma'^J_{ij} = \sigma_{ij}^J - \delta_{ij}\sigma_0^J$, where δ_{ij} is Kronecker delta.

We adopt an additive split of the deformation of rate into elastic deformation of rate and plastic deformation of rate [23, 24]. Stress-strain relationship for elastic deformation is written as

$$\Sigma'^J = 2G\mathbf{d}', \Sigma_0^J = 3K\mathbf{d}_0, \quad (2)$$

where $G = \frac{E}{2(1+\mu)}$, $K = \frac{E}{3(1-2\mu)}$, E – Young's modulus, μ – Poisson's ratio. In this case the elastic stress-strain relationships will satisfy to indifference principle.

4. Algorithm of computation

Process of deformation is represented as a sequence of equilibrium states [23–27]. Conversion from previous state to next one proceeds by increment load. On each step of a loading the geometry and stress state are updated. Techniques of computation is consist in a development of algorithm of calculation of the state $(l+1)$ when the state l is known. Consider the equation of the virtual powers in an actual configuration [31–33]

$$\iiint_{\Omega} \Sigma : \delta \mathbf{d} d\Omega = \iiint_{\Omega} \mathbf{Q} \cdot \delta \mathbf{v} d\Omega + \iint_{S^\sigma} \mathbf{P} \cdot \delta \mathbf{v} dS, \quad (3)$$

where Ω is the current volume, S^σ is the surface on which the force \mathbf{P} is applied, \mathbf{Q} is the body force vector. The basic equation is obtained from the equation (3) using follow relations

$$\frac{d}{dt} \iiint_{\Omega} \Sigma : \delta \mathbf{d} d\Omega = \iiint_{\Omega} \left[\dot{\Sigma} : \delta \mathbf{d} + \Sigma : \delta \dot{\mathbf{d}} + \frac{J}{J} \Sigma : \delta \mathbf{d} \right] d\Omega,$$

$$\frac{d}{dt} \iiint_{\Omega} \mathbf{Q} \cdot \delta \mathbf{v} d\Omega = \iiint_{\Omega} \left(\dot{\mathbf{Q}} + \mathbf{Q} \frac{J}{J} \right) \cdot \delta \mathbf{v} d\Omega,$$

$$\frac{d}{dt} \iint_{S^\sigma} \mathbf{P} \cdot \delta \mathbf{v} dS = \iint_{S^\sigma} \left[\dot{\mathbf{P}} + \mathbf{P} \frac{J}{J} \right] \cdot \delta \mathbf{v} dS,$$

where $\delta \dot{\mathbf{d}} = -\frac{1}{2}[\mathbf{h}^T \cdot \delta \mathbf{h}^T + \mathbf{h} \cdot \delta \mathbf{h}]$, $\frac{J}{J} = 3d_0 = \text{div} \mathbf{v}$, where \mathbf{v} – the velocity of a particle.

The von Mises yield criterion and radial return method are applied to find the current state [23, 25, 27]. Since the derived stress-strained state does not satisfy basic equation (3), then we carry out iteration procedure for increasing the accuracy current state. This procedure is based on the introduction of the “additional stresses” on virtual deformation of the rate. In this case the final equation will be

$$\begin{aligned} & \iiint_{\Omega_l} \left\{ 2G^l \mathbf{d}'^m : \delta^l \mathbf{d}' + 9K^l \mathbf{d}_0^m : \delta^l \mathbf{d}_0 + {}^l \Sigma : \left[\delta^l \mathbf{d} \cdot {}^l \boldsymbol{\omega}^m - {}^l \boldsymbol{\omega}^m \cdot \delta^l \mathbf{d} + \delta^l \dot{\mathbf{d}}^m \right] + \right. \\ & \left. + \frac{{}^l J^m}{{}^l J^m} {}^l \Sigma : \delta^l \mathbf{d} \right\} d\Omega - \iiint_{\Omega_l} \frac{{}^l J^m}{{}^l J^m} {}^l \mathbf{Q} \cdot \delta^l \mathbf{v} d\Omega - \iint_{S_l^\sigma} \frac{{}^l J^m}{{}^l J^m} {}^l \mathbf{P} \cdot \delta^l \mathbf{v} dS = \\ & = \iiint_{\Omega_l} {}^l \dot{\mathbf{Q}} \cdot \delta^l \mathbf{v} d\Omega + \iint_{S_l^\sigma} {}^l \dot{\mathbf{P}} \cdot \delta^l \mathbf{v} dS - \\ & - \left\{ \iiint_{\Omega_l} [{}^l \Sigma : \delta^l \mathbf{d} - {}^l \mathbf{Q} \cdot \delta^l \mathbf{v}] d\Omega + \iint_{S_l^\sigma} {}^l \mathbf{P} \cdot \delta^l \mathbf{v} dS \right\} \Delta t^{-1} + \iiint_{\Omega_l} {}^l \Sigma_\sigma^m \cdot \delta^l \mathbf{d} d\Omega, \end{aligned} \quad (4)$$

where m is the number of iteration, l is the number of step.

For quasi-static problems will take

$${}^l \mathbf{v}^m = \frac{{}^{l+1} \mathbf{R}^m - {}^l \mathbf{R}}{\Delta t} = \frac{\Delta {}^l \mathbf{u}^m}{\Delta t}.$$

And let $\Delta t=1$. In the result we get from (4), the equations for the increments of displacements $\Delta {}^l \mathbf{u}^m$. After solving the general system of equations (4), we define tensors ${}^l \mathbf{d}^m$, ${}^l \boldsymbol{\omega}^m$, using the relations (1, 2) find the rate of Cauchy stress ${}^l \dot{\boldsymbol{\Sigma}}^m$ and trial stress tensor

$$\begin{aligned} {}^{l+1} \boldsymbol{\Sigma}_0 &= {}^l \boldsymbol{\Sigma}_0 + {}^l \dot{\boldsymbol{\Sigma}}_0 \Delta t, \\ \tilde{\boldsymbol{\Sigma}}^m &= {}^l \boldsymbol{\Sigma}' + {}^l \dot{\boldsymbol{\Sigma}}^m \Delta t, \end{aligned}$$

check the yield condition

$$\tilde{\sigma}_i^m = \sqrt{\frac{3}{2} \tilde{\sigma}_{ij}^m \tilde{\sigma}_{ij}^m} \leq \sigma_T,$$

and if $\tilde{\sigma}_i^m > \sigma_T$ the radial return method with an iterative refinement of the current mode of deformation is applied:

$$\boldsymbol{\Sigma}'^{m+1} = \tilde{\boldsymbol{\Sigma}}^m \frac{\sigma_T}{\tilde{\sigma}_i^m}, \quad \boldsymbol{\Sigma}_\theta^{m+1} = \boldsymbol{\Sigma}'^{m+1} - \tilde{\boldsymbol{\Sigma}}^m. \quad (5)$$

Using relations (5) the general equation (4) is solved. If $\boldsymbol{\Sigma}_\theta^{m+1} \leq \text{tol}$ then the next configuration ${}^{l+1} \mathbf{y}^j = {}^l \mathbf{y}^j + \Delta {}^l \mathbf{u}^i$ and stress state ${}^{l+1} \boldsymbol{\Sigma} = \boldsymbol{\Sigma}^m$ are defined.

5. Numerical example

The numerical computational is based on eight node isoparametric hexahedron element [21].

The problem of bending of a rectangular cross-section beam is considered. The beam is rigidly fixed on the one side, and on the other loaded with a bending moment. The length of the beam $l=50$ cm, height $h=1$ cm, width $b=0.125$ cm, $E=1000$ kg/cm², Poisson's ratio is=0. The value of the moment at which the beam is bent into a ring, was taken from the analytical solution of the $M=2 \cdot \pi$ kg cm (figure 1).

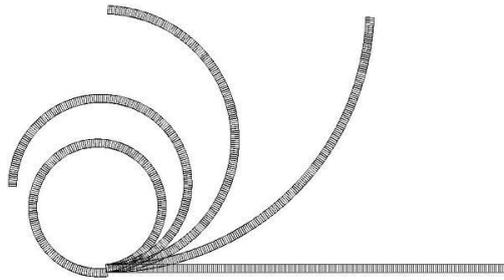


Figure 1. Different stages of the bending of a beam.

The problem of elastoplastic deformation rigidly clamped at both ends of the beam under the action of distributed load $q=27$ kg/cm². The length of the beam $l=25$ cm, height $h=1$ cm, width $b=0.125$ cm, $E=20000$ kg/cm², $\mu=0$, $\sigma_T = 750$ kg/cm² (figure 2, 3).

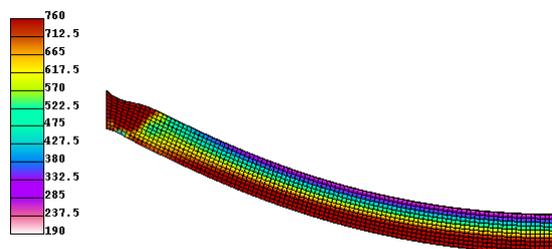


Figure 2. Equivalent stress on deformed structures.

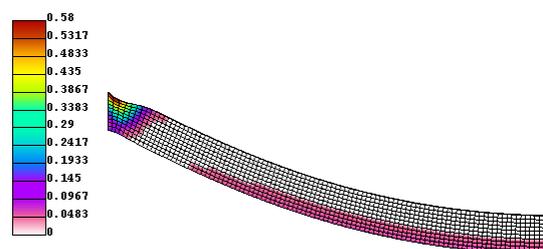


Figure 3. Equivalent plastic strain on deformed structures.

6. Conclusion

A method of numerical investigation of elastic-plastic solids with finite deformations is considered. For plasticity the von Mises criterion is applied. Incremental loading procedure is used, where allowing the variation equation is derived from the principle of virtual work in the current configuration. For the simulation of plastic deformation the radial return method with iterative refinement of the current stress-strain state is applied. The numerical discretization is based on the finite element method.

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