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Solution for the problem of the Stokes flow past a body of arbitrary shape using conformal mapping

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Abstract. An approach to the calculation of flow of an incompressible viscous fluid in the Stokes approximation is developed based on the finite difference method with the use of conformal mapping. The problems of flow around a circular cylinder and ellipse in a periodic circular cell have been solved. The method is particularly well suited for solving problems of flow around bodies of an arbitrary curved shape.

1. The problem statement

A flow around a body of an arbitrary shape in a circular periodic cell at low Reynolds numbers is considered (fig. 1, a). In the Stokes flow approximation, the dimensionless problem of a fluid flow in the domain *ABCD* is written in a stream function ψ – vorticity ω variables [1]:

$$\begin{aligned} \Delta\psi &= -\omega \\ \Delta\omega &= 0 \end{aligned} \tag{1}$$

On the outer boundary of the periodic cell with radius h (line *AD*), the Kuwabara conditions are adopted [2]

$$\omega = 0,\ \psi = y \tag{2}$$

On the surface of the body (line *BC*) no-slip conditions are applied

$$\psi = 0,\ \frac{\partial\psi}{\partial n} = 0 \tag{3}$$

On the lines *AB* and *CD* the symmetry conditions are taken

$$\psi = 0,\ \omega = 0 \tag{4}$$

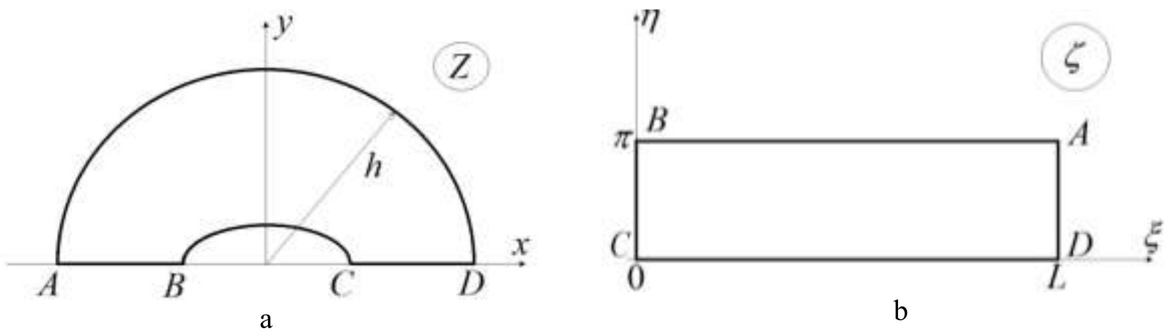


Figure. 1. Computational domain in the physical (a) and canonical (b) plane.

Let us introduce the function of conformal mapping of a rectangular area in a canonical plane $\zeta = \xi + i\eta$ (Fig.1, b) to the region of flow in the physical plane $z = x + iy$ (Fig.1, a)

$$z = \sum_{k=-N}^{N} a_k e^{k\zeta} \tag{5}$$

where a_k are the real coefficients that are found from the numerical solution of the problem of conformal mapping [3]. The size of the rectangular domain is $0 \le \eta \le \pi,\ 0 \le \xi \le L$ where the value L is determined in the course of the conformal mapping. Note that for the domain of an arbitrary shape the Laurent series (5) can be infinite but we use finite but large value N (the number of members of the Laurent series) to provide needed accuracy of conformal mapping.

Dividing the real and imaginary parts in (5) we obtain:

$$\begin{aligned} x &= \sum a_k e^{k\xi} \cos k\eta \\ y &= \sum a_k e^{k\xi} \sin k\eta \end{aligned} \tag{6}$$

It is known that in curvilinear coordinate system the Laplace operator has the form

$$\Delta\psi = \frac{1}{H_1 H_2}\left(\frac{\partial}{\partial\xi}\left(\frac{H_1}{H_2}\frac{\partial\psi}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{H_2}{H_1}\frac{\partial\psi}{\partial\eta}\right)\right) \tag{7}$$

where Lame coefficients are expressed as

$$H_1 = \left(\left(\frac{\partial x}{\partial\xi}\right)^2 + \left(\frac{\partial y}{\partial\xi}\right)^2\right)^{\frac{1}{2}},\ H_2 = \left(\left(\frac{\partial x}{\partial\eta}\right)^2 + \left(\frac{\partial y}{\partial\eta}\right)^2\right)^{\frac{1}{2}}$$

The Cauchy–Riemann conditions $\partial x / \partial\xi = \partial y / \partial\eta$, $\partial x / \partial\eta = -\partial y / \partial\xi$ are for the functions (6). In this case $H_1 = H_2$ and operator (7) has the form

$$\Delta_{x,y}\psi = H^{-1}\Delta_{\xi,\eta}\psi$$

where $H = \alpha^2 + \beta^2$, $\alpha = \partial x / \partial\xi = \sum k a_k e^{k\xi} \cos k\eta$, $\beta = \partial x / \partial\eta = -\sum k a_k e^{k\xi} \sin k\eta$

After transformation from the physical coordinates x, y to canonical ξ, η ones the equations (1) is written as:

$$\begin{aligned} \Delta\psi &= -\omega \cdot H(\xi,\eta) \\ \Delta\omega &= 0 \end{aligned} \tag{8}$$

The equations (8) are supplemented by the boundary conditions that correspond to the conditions (2–4). On the left side BC of the rectangle

$$\psi = 0,\ \partial\psi / \partial\xi = 0 \tag{9}$$

On the right side *AD*

$$\omega = 0,\ \psi = \sum a_k e^{k\xi} \sin k\eta \tag{10}$$

On the top and bottom sides *CD, AB*

$$\omega = 0,\ \psi = 0 \tag{11}$$

To solve Eq. (8) with the boundary conditions (9–11) by finite difference method we introduce the meshing of computational domain *ABCB* with nodes (ξ_j, η_i) so that $\xi_j = jh_\xi$, $j = \overline{0,M}$, $\eta_i = ih_\eta$, $i = \overline{0,N}$. We determine the discrete values of the functions $\psi_j^i = \psi(\xi_j, \eta_i)$, $\omega_j^i = \omega(\xi_j, \eta_i)$ in the nodes. Approximating the partial derivatives of order 2 by central differences, we can write Eq. (8) in a discrete form

$$\frac{\psi_{j-1}^i - 2\psi_j^i + \psi_{j+1}^i}{h_\xi^2} + \frac{\psi_j^{i-1} - 2\psi_j^i + \psi_j^{i+1}}{h_\eta^2} = -\omega_j^i \cdot H_j^i \tag{12}$$

$$\frac{\omega_{j-1}^i - 2\omega_j^i + \omega_{j+1}^i}{h_\xi^2} + \frac{\omega_j^{i-1} - 2\omega_j^i + \omega_j^{i+1}}{h_\eta^2} = 0 \tag{13}$$

where $H_j^i = H(\xi_j, \eta_i)$. The boundary conditions (9–11) on lines *BC*, *AD, AB* , *CD* are written as:

$$\psi_0^i = 0,\ \left(\psi_1^i - \psi_0^i\right)/h_\xi = 0 \tag{14}$$

$$\omega_M^i = 0,\ \psi_M^i = \sum a_k e^{kL} \sin k\eta_i \tag{15}$$

$$\omega_j^N = 0,\ \psi_j^N = 0 \tag{16}$$

$$\omega_j^0 = 0, \psi_j^0 = 0 \tag{17}$$

To obtain the system of algebraic equations (SAE) for ψ_j^i, ω_j^i Eq. (10–11) are written in the form

$$\frac{1}{h_\xi^2}\psi_{j-1}^i + \frac{1}{h_\xi^2}\psi_{j+1}^i - \left(\frac{2}{h_\xi^2} + \frac{2}{h_\eta^2}\right)\psi_j^i + \frac{1}{h_\eta^2}\psi_j^{i-1} + \frac{1}{h_\eta^2}\psi_j^{i+1} = -\omega_j^i \cdot H_j^i \tag{18}$$

$$\frac{1}{h_\xi^2}\omega_{j-1}^i + \frac{1}{h_\xi^2}\omega_{j+1}^i - \left(\frac{2}{h_\xi^2} + \frac{2}{h_\eta^2}\right)\omega_j^i + \frac{1}{h_\eta^2}\omega_j^{i-1} + \frac{1}{h_\eta^2}\omega_j^{i+1} = 0 \tag{19}$$

Taking into account the conditions (14–17) we will obtain SAE of order $4\times((M\text{–}2)\times(N\text{–}2))^2$. The direct solution of the SAE requires more memory but is more effective than the separate solution of (18) и (19) and use of iterations for the converged joint solution.

2. Results and discussion

As a test the flow around the cylinder in a periodic cell with the Kuwabara conditions is solved. The problem has also an analytical solution [2]

$$\psi^a(r,\theta) = \left(A\frac{1}{r} + Br + Cr\ln r + Dr^3\right)\sin\theta \tag{20}$$

$$\omega^a(r,\theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi^a}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\frac{\partial^2 \psi^a}{\partial \theta^2} = -2\left(\frac{C}{r} + 4Dr\right)\sin\theta \tag{21}$$

where $A = 0.5\,(1-\alpha/2)/k$, $B = -0.5\,(1-\alpha)/k$, $C = 1/k, D = -0.25\alpha/k$, $\alpha = h^{-2}$, $k = \alpha - 0.25\alpha^2 - 0.5\ln\alpha - 0.75$, $r = \sqrt{x^2+y^2}$, $\theta = \mathrm{arctg}(y/x)$.

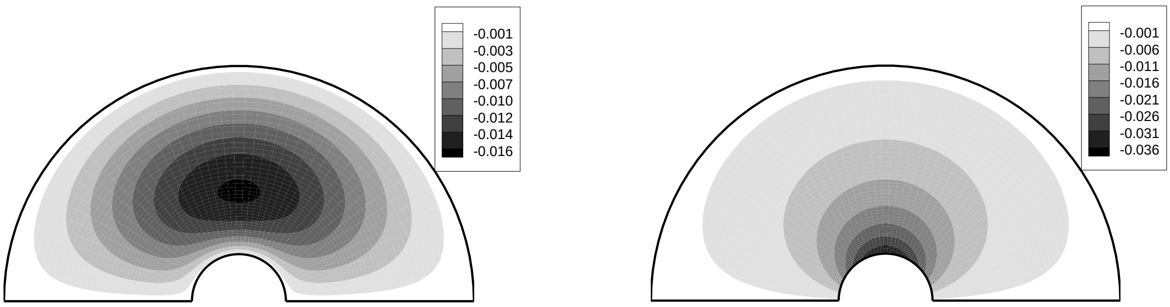
A numerical solution for $M=N=100$ is compared with analytical formulas (20–21). To estimate the accuracy of the developed method the absolute

$$E_\psi = \psi^a(x,y) - \psi(x,y),\ E_\omega = \omega^a(x,y) - \omega(x,y)$$

and relative errors

$$\varepsilon_\psi = \frac{\max\left|E_\psi(x,y)\right|}{\max\left|\psi^\circ(x,y)\right|},\ \varepsilon_\omega = \frac{\max\left|E_\omega(x,y)\right|}{\max\left|\omega^\circ(x,y)\right|}$$

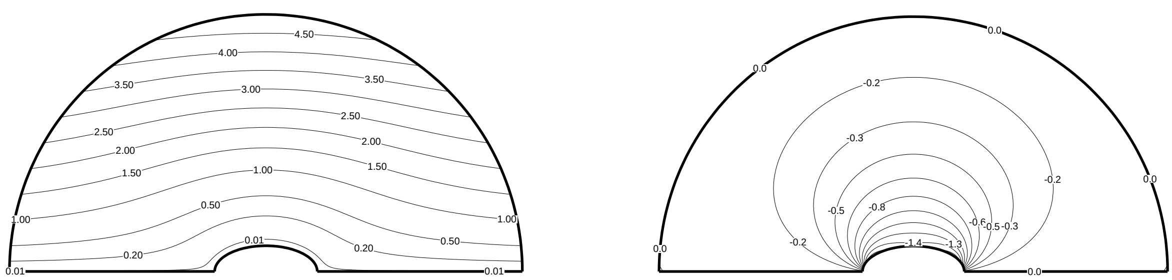
were calculated. The distributions of the quantities $E_\psi(x,y)$ и $E_\omega(x,y)$ are shown in fig.2,a and 2,b respectively. The values of relative error are $\varepsilon_\psi = 0.003267$, $\varepsilon_\omega = 0.017285$. The conclusion can be made that the developed method gives high accuracy.



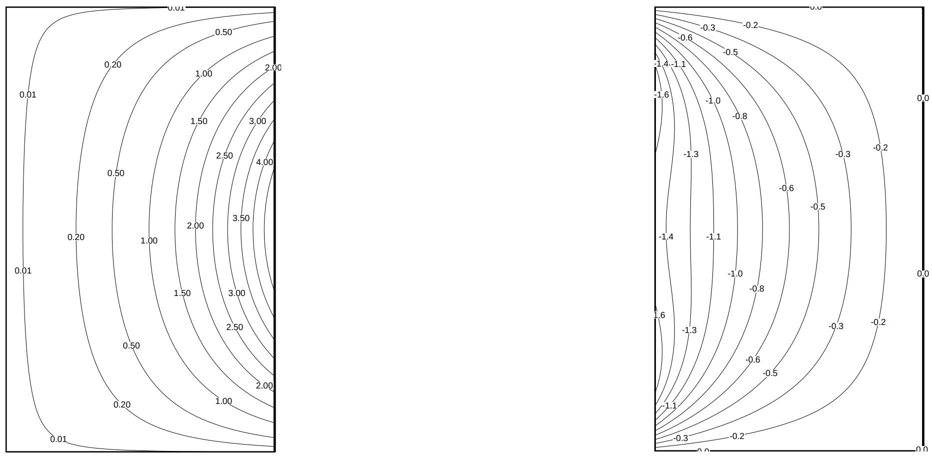
a b

Figure 2. The distributions of $E_\psi(x,y)$, $E_\omega(x,y)$

The problem of fluid flow around an elliptical cylinder with half-axes $a_1 = 0.5$, $a_2 = 1$ and radius of outer boundary h=5 has been solved. Examples of calculated streamlines and vorticity isolines in the physical and canonical planes are given in fig.3.



a b



c d

Figure 3. The streamlines (a, c) and vorticity (b, d) isolines in the physical (a, b) and canonical (c, d) planes.

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