# Mathematical modelling of the mass transfer process between two coaxial cylinders in the problem of thermal creep

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**Abstract.** An analytic solution to the Williams equation in the thermal creep flow problem in the channel, formed by two coaxial cylinders, has been constructed using the kinetic approach. For the boundary condition on the channel walls, the diffusion model is used. Taking into account the constructed distribution function for various values of the ratio of radii of cylinders and Knudsen number, the value of the mass flow rate in channel is calculated. The analysis of the obtained results during the transition to free molecular and hydrodynamic regimes is done.

## 1. Introduction

From the point of view of applications, one of the most important problems in the dynamics of a rarefied gas is the study of gas flow in a channel [1]. In the wake of the development of micro and nanotechnologies, considerable attention has been paid recently to analysis of gas flows in channels with arbitrary cross sections. For example, the flow of a rarefied gas in a channel, formed by two coaxial cylinders, in a channel with constant rectangular and elliptical cross sections was considered in [2-4], respectively. In [5], the problem of mass and heat transfer was viewed in a cylindrical channel with constant pressure and temperature gradients by using Shakhov's model of Boltzmann kinetic equation for cylindrical channels. In [6] the profiles of the mass flow velocity of the gas and the heat flux vector in the channel in free molecular regime have been constructed. A more comprehensive analysis of the current state of the issue on the study of rarefied gas flows in channels one can found in [7]. The aim of this work is to construct an analytic solution to the problem of the thermal creep flow in the channel, formed by two coaxial cylinders under the action of a small thermal pressure drop by using the Williams equation. In this work, for the boundary conditions on the channel walls, we used diffuse reflection model. As a supplement, we constructed a profile of the gas mass velocity in the channel for various values of the ratio of radii of cylinders and Knudsen number.

## 2. Statement of the problem and mathematical model

Let us consider a channel, formed by two coaxial cylinders with radii $R_1^{'}$ and $R_2^{'}$ ($R_2^{'} > R_1^{'}$). We assume that a constant temperature gradient is maintained in the channel along the $Oz'$ axis. We use the diffuse reflection model for the boundary condition at the channel walls. We denote by $\mathbf{r}'$ the radius vector of gas molecules, by $\rho'$, $\varphi$, $z'$ their cylindrical coordinates in configuration space, and by $\mathrm{v}_\rho, \mathrm{v}_\varphi, \mathrm{v}_z$ - projection of velocity vector $\mathbf{v}$ on the coordinate axes.

Mass flux through the channel cross section we define according to [7] as

$$J'_M = 2\pi m n \int_{R'_1}^{R'_2} u'_z(\rho')\rho' d\rho' .$$

Hear $m$ and $n$ are the mass and concentration of the gas molecules, $u'_z(\rho')$ – $z'$-component of the mass velocity of gas.

The mass velocity of gas we obtained proceeding from the statistical meaning of the distribution function $f(\mathbf{r}',\mathbf{v})$ [8]

$$u_z(\rho') = \frac{1}{n(z')}\int \mathrm{v}_z f(\mathbf{r}',\mathbf{v}) d^3\mathbf{v} .$$

For the sake of convenience in further computations, we pass in previous expressions to the dimensionless variable, by considering $\rho = \rho'/R'_2$, $z = z'/R'_2$ as the coordinates of the dimensionless radius vector; $\mathbf{C} = \beta^{1/2}\mathbf{v}$ as the dimensionless velocity of gas molecules, there $\beta = m/(2k_B T_0)$, $k_B$ is the Boltzmann constant, $T_0$ is the gas temperature at a point chosen as the origin of coordinates. As a result, we obtain the following expressions for dimensionless mass velocity and a mass flux:

$$U_z(\rho) = \pi^{-3/2}\int C_z \exp(-C^2) f(\mathbf{r},\mathbf{C}) d^3\mathbf{C} , \tag{1}$$

$$J_M = \frac{4}{(1-R_1^2)}\int_{R_1}^{1} U_z(\rho)\rho\, d\rho \tag{2}$$

The distribution function of gas molecules $f(\mathbf{r}',\mathbf{v})$ we obtain from Williams equation, which in the chosen coordinate system can be written in the form [9]

$$\mathrm{v}_\rho \frac{\partial f}{\partial \rho'} + \frac{\mathrm{v}_\varphi}{\rho'}\frac{\partial f}{\partial \varphi} + \mathrm{v}_z \frac{\partial f}{\partial z'} + \frac{\mathrm{v}_\varphi^2}{\rho'}\frac{\partial f}{\partial \mathrm{v}_\rho} - \frac{\mathrm{v}_\rho \mathrm{v}_\varphi}{\rho'}\frac{\partial f}{\partial \mathrm{v}_\varphi} = \frac{\omega}{\gamma l_g}(f_* - f) , \tag{3}$$

$$\omega = |\,\mathbf{v} - \mathbf{u}(\mathbf{r}')\,|, \qquad f_* = n_*\left(\frac{m}{2\pi k_B T_*}\right)^{3/2} \exp\left(-\frac{m}{2k_B T_*}(\mathbf{v}-\mathbf{u}_*)^2\right), \qquad \gamma = 5\sqrt{\pi}/4 .$$

Here quantities $n_*$, $\mathbf{u}_*$ and $T_*$, entering into $f_*$, are not the local density, temperature, and velocity, but are certain parameters that are chosen from the condition that the model collision integral satisfies the laws of conservation of the number of particles, momentum, and energy [9]. For the diffuse reflection model for the boundary condition at the channel walls we can write

$$f^+(\mathbf{r}'_s,\mathbf{v}) = f_s(\mathbf{r}',\mathbf{v}), \quad \mathbf{n}_i\mathbf{v} > 0,\ i = 1,2 . \tag{4}$$

Here $f^+(\mathbf{r}'_s,\mathbf{v})$ – distribution function of gas molecules, reflected from the surface of the channel, $f_s(\mathbf{r}',\mathbf{v})$ – local equilibrium distribution function with the parameters on the streamlined gas cylindrical surfaces, $\mathbf{n}_1$ and $\mathbf{n}_2$ are the normal vectors to the surfaces of the cylinders, which are directed into the gas,

$$f_s(\mathbf{r}',\mathbf{v}) = n(z')\left(\frac{m}{2\pi k_B T(z')}\right)^{3/2} \exp\left(-\frac{m}{2k_B T(z')}\mathbf{v}^2\right). \tag{5}$$

We assume that a relative temperature drop over the mean free path of gas molecules is small. Then, the problem allows linearization, and the coordinate and velocity distribution function of gas molecules can be written in the form

$$f(\mathbf{r}',\mathbf{v}) = f_0(C)(1 + h(\mathbf{r},\mathbf{C}) + C_z Z_0(\rho,\mathbf{C})) , \tag{6}$$

$$h(\mathbf{r},\mathbf{C}) = 2C_z U_0 + G_T\left[\left(z - \gamma Kn\frac{C_z}{C}\right)\left(C^2 - \frac{5}{2}\right) - \frac{2\gamma Kn}{3\sqrt{\pi}}C_z\right].$$

Here $f_0(C) = n_0(\beta/\pi)^{3/2}\exp(-C^2)$ is absolute Maxwellian, $G_T$ is the dimensionless temperature gradient, and $Z_0(\rho,\mathbf{C})$ is the linear correction to the local equilibrium distribution function, which takes into account the influence of the walls, $Kn = l_g / R_2'$ is Knudsen number, $l_g = \eta_g\,\beta^{-1/2}/p$ is the mean free path of the gas molecules.
Substituting (6) into (3), we arrive at the equation for finding function $Z_0(\rho,\mathbf{C})$:

$$\left(\cos\psi\frac{\partial Z_0}{\partial\rho} - \frac{\sin\psi}{\rho}\frac{\partial Z_0}{\partial\psi}\right)\gamma Kn\sin\theta + Z_0(\rho,\mathbf{C}) = \frac{3}{4\pi}\int C'\exp(-C'^2)C_z'^2 Z_0(\rho,\mathbf{C}')d^3\mathbf{C}'. \quad (7)$$

For the sake of convenience in further computations, we pass in (7) to the spherical coordinate system in the velocity space, assuming that $C_\rho = C\cos\psi\sin\theta$, $C_\varphi = C\sin\psi\sin\theta$, $C_z = C\cos\theta$, where the angles ψ and θ are measured from the positive directions of the axes $C_\rho$ and $C_z$. The boundary conditions by taking into account (4), (5) we rewrite as follows

$$Z_0(\rho,\mathbf{C})\big|_s = -2U_0 + \gamma KnG_T\left(C - \frac{5}{2C}\right) + \frac{2\gamma KnG_T}{3\sqrt{\pi}},\ \mathbf{n}_1\mathbf{C} > 0,\ \mathbf{n}_2\mathbf{C} > 0. \quad (8)$$

We seek a solution of (7) in the form of decomposition in two orthogonal functions $e_1(C) = 1$ and $e_2(C) = C - 5/(2C)$:

$$Z_0(\rho,\varphi,\mathbf{C}) = Z_1(\rho,\psi,\sin\theta) + \left(C - \frac{5}{2C}\right)\gamma KnG_T[Z_2(\rho,\psi,\sin\theta) + 1]. \quad (9)$$

The orthogonally here is understood as the equality to zero of the integral $\int\limits_0^{+\infty} e_1(C)e_2(C)C^5\exp(-C^2)dC$. Substituting (9) into (7), we arrive at the following system of equations

$$\gamma Kn\left(\cos\psi\frac{\partial Z_1}{\partial\rho} - \frac{\sin\psi}{\rho}\frac{\partial Z_1}{\partial\psi}\right)\sin\theta + Z_1(\rho,\psi,\sin\theta) =$$

$$= \frac{3}{4\pi}\int\limits_0^{\pi}\cos^2\theta'\sin\theta' d\theta'\int\limits_0^{2\pi} Z_1(\rho,\psi',\sin\theta')d\psi', \quad (10)$$

$$\gamma Kn\left(\cos\psi\frac{\partial Z_2}{\partial\rho} - \frac{\sin\psi}{\rho}\frac{\partial Z_2}{\partial\psi}\right)\sin\theta + Z_2(\rho,\psi,\sin\theta) + 1 = 0, \quad (11)$$

with boundary conditions

$$Z_1(\rho,\psi,\sin\theta)\big|_s = -2U_0 + \frac{2\gamma KnG_T}{3\sqrt{\pi}},\quad \mathbf{n}_1\mathbf{C} > 0,\ \mathbf{n}_2\mathbf{C} > 0, \quad (12)$$

$$Z_2(\rho,\psi,\sin\theta)\big|_s = 0,\quad \mathbf{n}_1\mathbf{C} > 0,\quad \mathbf{n}_2\mathbf{C} > 0. \quad (13)$$

It is easy to see from direct substitution that the solution (10) with boundary conditions (12) has the form:

$$Z_1(\rho,\psi,\sin\theta) = -2U_0 + \frac{2\gamma KnG_T}{3\sqrt{\pi}}. \tag{14}$$

From (13) we can see, that the boundary conditions for $Z_2(R_1,\psi,\sin\theta)$ are homogeneous:

$$Z_2(R_1,\psi,\sin\theta) = 0, \quad \cos\psi < 0, \tag{15}$$

$$Z_2(1,\psi,\sin\theta) = 0, \quad \cos\psi > 0. \tag{16}$$

Here $R_1 = R'_1 / R'_2$.

The solution of equation (11) with boundary conditions (15) and (16) is sought by the method of characteristics [10]. The system of equations for characteristic of equation (11) has the form:

$$\frac{d\rho}{\gamma Kn\sin\theta\cos\psi} = -\frac{\rho d\psi}{\gamma Kn\sin\theta\sin\psi} = -\frac{dZ_2}{Z_2(\rho,\psi,\sin\theta)+1} = dt \tag{17}$$

Integrating this system of equations, we obtain the two first independent integrals:

$$\rho|\sin\psi| = C_1, \qquad (Z_2(\rho,\psi,\sin\theta)+1)\exp\left(\frac{\rho\cos\psi}{\gamma Kn\sin\theta}\right) = C_2. \tag{18}$$

The integration constants $C_1$ and $C_2$ in (18) we exclude using boundary conditions (15) and (16). Thus we obtain

$$Z_2(\rho,\psi,\sin\theta) = \exp(-t_k) - 1, \; k = 1,2.$$

The value of the variable $t$ we determine from the condition that the reflection of molecules from the surface of the inner cylinder occurs at $-\alpha \le \psi \le \alpha$:

$$t_1 = \frac{\rho\cos\psi - \sqrt{R_1^{\,2} - \rho^2\sin^2\psi}}{\gamma Kn\sin\theta}, \; \alpha = \arccos\frac{\sqrt{\rho^2 - R_1^{\,2}}}{\rho};$$

and at $\alpha \le \psi \le 2\pi - \alpha$ reflection occurs from the surface of the outer cylinder:

$$t_2 = \frac{\rho\cos\psi + \sqrt{1 - \rho\sin\psi^2}}{\gamma Kn\sin\theta}.$$

Substituting the distribution function (6) in (1) and taking into account the obtained results, we find dimensionless mass velocity of gas in the channel

$$U_z(\rho) = \frac{G_T\gamma Kn}{3\sqrt{\pi}}\left(1 - \frac{3}{2\pi}\int_0^{\pi}\cos^2\theta\sin\theta\,d\theta\left(\int_0^{\alpha}\exp(-t_1)d\psi + \int_{\alpha}^{\pi}\exp(-t_2)d\psi\right)\right). \tag{19}$$

The values of $J_M$ that we found according to (2) using the computer algebra system Maple 17 under different Knudsen number and relationship of the radii of the cylinders $R_1'$ and $R_2'$ are given in table 1.

**Table 1.** Values of $J_M$ for various ratios $R_1'/R_2'$.

| $Kn$ | $R_1 = R_1'/R_2'$ | | | | | |
|---|---|---|---|---|---|---|
| | 0 | 0 [5] | 0.01 | 0.1 | 0.5 | 0.9 |
| 0.0001 | 0.0001 | - | 0.0001 | 0.0001 | 0.0001 | 0.0001 |
| 0.0010 | 0.0008 | - | 0.0008 | 0.0008 | 0.0008 | 0.0008 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 0.0100 | 0.0083 | 0.0116 | 0.0083 | 0.0083 | 0.0082 | 0.0076 |
| 0.1000 | 0.0764 | 0.1020 | 0.0764 | 0.0757 | 0.0698 | 0.0414 |
| 0.5000 | 0.2705 | 0.3027 | 0.2695 | 0.2601 | 0.2014 | 0.0761 |
| 1.0000 | 0.3881 | 0.3968 | 0.3862 | 0.3684 | 0.2660 | 0.0878 |
| 2.0000 | 0.4977 | 0.4784 | 0.4948 | 0.4677 | 0.3208 | 0.0963 |
| 5.0000 | 0.6080 | - | 0.6040 | 0.5666 | 0.3724 | 0.1034 |
| 10.000 | 0.6632 | 0.6280 | 0.6586 | 0.6158 | 0.3970 | 0.1065 |
| 100.00 | 0.7376 | 0.7210 | 0.7321 | 0.6816 | 0.4290 | 0.1103 |
| 1000.0 | 0.7502 | 0.7493 | 0.7446 | 0.6927 | 0.4342 | 0.1108 |
| 10000 | 0.7520 | - | 0.7463 | 0.6943 | 0.4349 | 0.1109 |

## 3. Results and discussion

Profiles of the $z$-components of the mass velocity in the channel, calculated according to (19), are given on figures 1 and 2. The case $R_1 = 0$ is described the processes of mass transfer through the cross section of round pipe without internal cylinder.

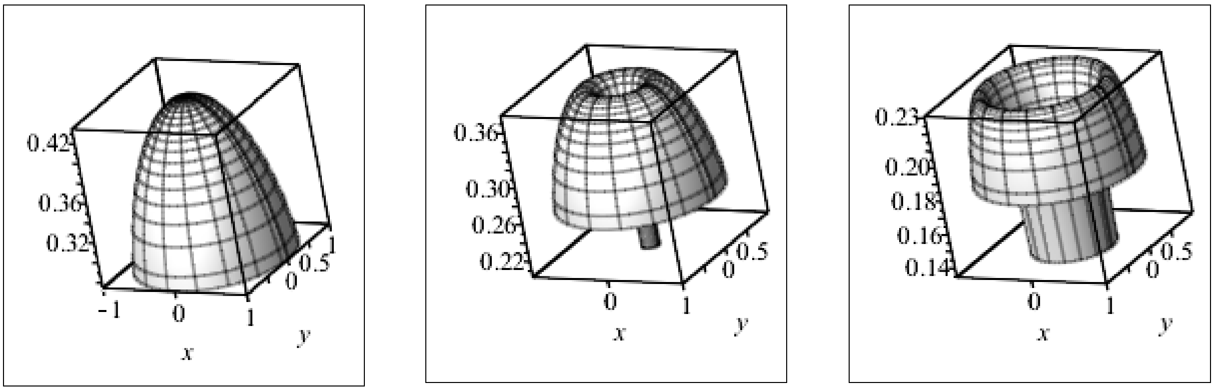
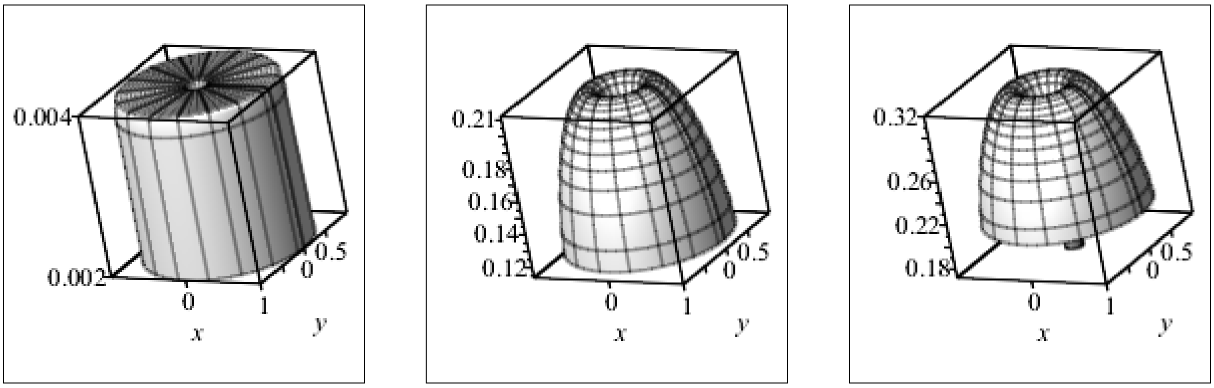



**Figure 1.** $U_z(\rho)$ for $Kn = 100$
$R_1 = 0$, 0.1 and 0.5 (from left to right)..

From the presented figures one can see that at $Kn >> 0$ the distribution of mass velocity has a maximum, which shifts toward the inner cylinder with the decrease of its radius. When $R_1 << 1$ and $Kn >> 1$ the distribution of the velocity profile close to a paraboloid of rotation.



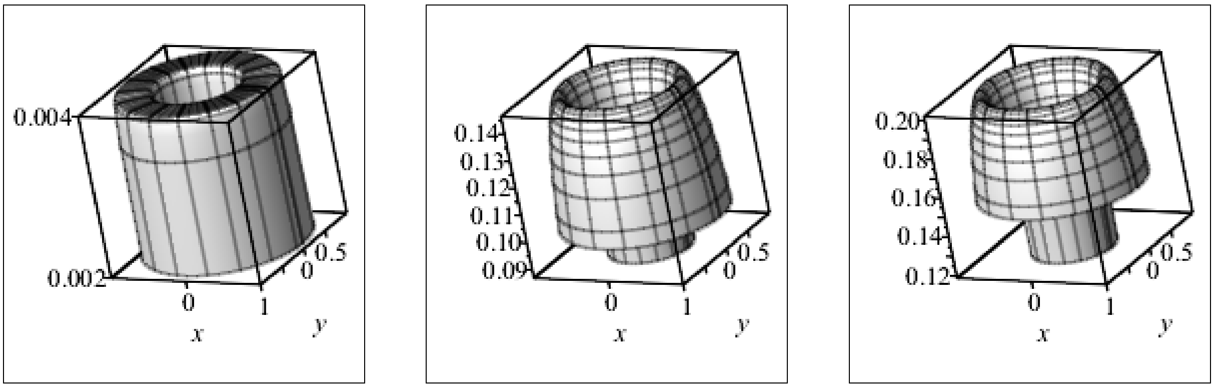



**Figure 2**. $U_z(\rho)$ for $R_1 = 0.1$ (top row) and $R_1 = 0.5$ (bottom row)
for $Kn = 0.01$, 1 and 5 (from left to right)..

For the close to the free molecular flow regime expression (19) for the mass velocity we can to decompose in a series in the small parameter $1/Kn$. In this case, by limited to the linear terms of the expansion, according to (2), we find

$$J_M = \frac{G_T}{\pi^{1/2}(1-R_1{}^2)} \int_{R_1}^{1} \left( \int_{\alpha}^{\pi} \sqrt{1-\rho^2 \sin^2\psi}\, d\psi - \int_{0}^{\alpha} \sqrt{R_1^2 - \rho^2 \sin^2\psi}\, d\psi \right) \rho d\rho \, . \tag{20}$$

The expression (20) defines a specific mass flow through the channel with cross-section formed by two coaxial cylinders, in a free molecular flow regime, under the effect of the longitudinal temperature gradient. For the case $R_1 = 0$ integrals in (20) may be calculated analytically. In this case $J_M / G_T = 4/(3\sqrt{\pi})$, that coincides with the result of the cylindrical channel [7]. For flow regimes close to the hydrodynamic regime, analysis of the expression (19) gives the following result

$$J_M = \frac{5G_T Kn}{6} . \tag{21}$$

Thus, for the regime close to the hydrodynamic, specific mass flux does not depend on the radii of the cylinders. The last statement is confirmed by the results presented in table 1 for $Kn \le 0.001$. From (21) one can obtained, that the thermal slip coefficient is equal $5/6$ and it is considering with the corresponding result, obtained in [11].

The results of calculations for the specific mass flow for $0.0001 \le Kn \le 10000$ and $R_1 = 0.00$, 0.01, 0.10, 0.50 and 0.90 are presented in table 1. For comparison, let us consider the values obtained in [5] using Shakhov's model of Boltzmann kinetic equation for cylindrical channels. The difference between the values in table 1 we can explained by the fact, that the macro parameters of gas in the problem on thermal creep strongly depend on the choice of model of collision integral [7].

## 4. Conclusions

In this work, we have calculated the gas mass flux in a channel, formed by two coaxial cylinders. The profiles of the mass flow velocity of the gas in the channel have been constructed. For different values of Knudsen number and various ratios of cylinder radii the specific gas mass flux through the channel cross section have been calculated. The analysis of the obtained expressions is done. It is shown that in extreme cases then $Kn \ll 1$ and $Kn \gg 1$, the results obtained in this work are moving in similar results for the hydrodynamic and free-regimes, respectively.

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