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A minimization method on the basis of embedding the feasible set and the epigraph

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Abstract. We propose a conditional minimization method of the convex nonsmooth function which belongs to the class of cutting-plane methods. During constructing iteration points a feasible set and an epigraph of the objective function are approximated by the polyhedral sets. In this connection, auxiliary problems of constructing iteration points are linear programming problems. In optimization process there is some opportunity of updating sets which approximate the epigraph. These updates are performed by periodically dropping of cutting planes which form embedding sets. Convergence of the proposed method is proved, some realizations of the method are discussed.

1. Introduction
Some algorithms from a class of cutting methods (e. g. [1 — 9]) are used to solve many applied optimization problems. Examples of these problems which are successfully solved by the mentioned methods can be found in [10 — 12]. Cutting methods are chosen for solving optimization problems, because there are some possibilities of estimating the proximity of the value of the objective function at the current iteration point to the optimal value.

Usually during practical implementations of the cutting methods there is some problem of accumulating cutting planes which form approximating sets. In this work the proposed method belongs to the mentioned class of methods and is characterized by using operation of simultaneous approximating the epigraph and the feasible set. The main feature of the method is possibility of updating embedding sets due to discarding any cutting planes.

2. Problem Settings
A problem

$$min\{ f(x) : x \in D\}, \tag{1}$$

is solved by the proposed method, where $f(x)$ is a convex function defined in n-dimensional Euclidian space R_n, and a set $D \subset R_n$ is closed and convex.

Suppose $f^* = \min\{f(x): x \in D\} > -\infty$, $X^* = \{x \in D : f(x) = f^*\}$, $x^* \in X^*$,

$$epi\ (f, R_n) = \{(x, \gamma) \in R_{n+1} : x \in R_n, \gamma \geq f(x)\},$$

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and let $int\, Q$ be an interior of the set Q, $W(\bar{u}, Q)$ be a cone of the normalized generalized-support vectors at the point $\bar{u}$ for the set Q, assign $K = \{0, 1, \dots\}$.

3. Minimization Method

The proposed method constructs an auxiliary sequence $\{\boldsymbol{u_i}\}$, $\boldsymbol{i \in K}$, and a base sequence of approximations $\{\boldsymbol{x_k}\}$, $\boldsymbol{k \in K}$, for solving problem (1) by the following rule. Select points

$$\boldsymbol{v' \in int\, D, \quad v'' \in int\, epi\, (f, R_n)}.$$

Choose a convex closed bounded set $\boldsymbol{M_0 \subset R_n}$ and a convex closed set $\boldsymbol{G_0 \in R_{n+1}}$ such that

$$\boldsymbol{x^* \in M_0, \quad epi(f, R_n) \subset G_0}.$$

Define numbers $\boldsymbol{\bar{\gamma}}$, $\boldsymbol{\varepsilon_k}$, $\boldsymbol{k \in K}$, according to conditions

$$\boldsymbol{\bar{\gamma} \leq min\{ f(x) : x \in M_0\}},$$

$$\boldsymbol{\varepsilon_k > 0,\ k \in K,\ \varepsilon_k \to 0,\ k \to \infty}. \tag{2}$$

Assign $\boldsymbol{i = 0}$, $\boldsymbol{k = 0}$.

1. Find a solution $\boldsymbol{u_i = (y_i, \gamma_i)}$ of the following problem

$$\boldsymbol{\min\{ \gamma : (x, \gamma) \in G_i, x \in M_k, \gamma \geq \bar{\gamma}\}}, \tag{3}$$

where $\boldsymbol{y_i \in R_n}$, $\boldsymbol{\gamma_i \in R_1}$.

2. Find $\boldsymbol{\bar{u}_i = (\bar{y}_i,\ \bar{\gamma}_i) \in R_{n+1}}$ as an intersection point of the segment $[\boldsymbol{v''}, \boldsymbol{u_i}]$ with the border of the set $\boldsymbol{epi\ (f, R_n)}$. If

$$\boldsymbol{\bar{u}_i = u_i, \quad y_i \in D}, \tag{4}$$

then $\boldsymbol{y_i \in X^*}$, and the process of solving problem (1) is finished.

3. If the inequality

$$\boldsymbol{||\bar{u}_i - u_i|| > \varepsilon_k} \tag{5}$$

is defined, then assign

$$\boldsymbol{G_{i+1} = S_i \cap \{u \in R_{n+1} : \langle b_i, u - \bar{u}_i \rangle \leq 0\}}, \tag{6}$$

where

$$\boldsymbol{b_i \in W\big(\bar{u}_i, epi(f, R_n)\big)}, \tag{7}$$

$$\boldsymbol{S_i = G_i}, \tag{8}$$

and increment $\boldsymbol{i}$ by one, go to Step 1. Otherwise go to Step 4.

4. Assign $\boldsymbol{i_k = i}$,

$$\boldsymbol{x_k = y_{i_k},\ \ \sigma_k = \gamma_{i_k}}, \tag{9}$$

and construct a set $\boldsymbol{G_{i+1}}$ in accordance with (6), (7), where $\boldsymbol{S_i}$ is a convex closed set such that

$$\boldsymbol{epi(f, R_n) \subset S_i}. \tag{10}$$

5. If $\boldsymbol{x_k \in D}$, then assign $\boldsymbol{M_{k+1} = M_k}$. Otherwise find $\boldsymbol{\bar{x}_k}$ as a point situated in the intersection

of the segment $[\boldsymbol{v', x_k}]$ with the border of the set $\boldsymbol{D}$, and assign

$$\boldsymbol{M_{k+1} = M_k \cap \{x \in R_n : \langle a_k, x - \bar{x}_k \rangle \leq 0\}},$$

where $\boldsymbol{a_k \in W(\bar{x}_k, D)}$.

6. Increment $\boldsymbol{i}$ and $\boldsymbol{k}$ by one, and go to Step 1.

Lets make some remarks about the proposed method. Firstly, prove stopping criteria of the method which is represented at Step 2.

Theorem 1. Suppose that expressions (4) is defined for some number $\boldsymbol{i \in K}$. Then the point $\boldsymbol{y_i}$ is a solution of problem (1).

Proof. In accordance with ways of choosing sets $\boldsymbol{M_0}$, $\boldsymbol{G_0}$, condition (10) and approach of constructing cutting planes it is easy to prove by induction that the feasible set of problem (3) contains the point $(\boldsymbol{x^*, f^*})$ for all $\boldsymbol{i \in K}$, $\boldsymbol{k \in K}$. Consequently, the solution $(\boldsymbol{y_i, \gamma_i})$ of this problem satisfies the inequality

$$\boldsymbol{\gamma_i \leq f^*} \tag{11}$$

for any $\boldsymbol{i \in K}$, $\boldsymbol{k \in K}$.

Assume that the number $\boldsymbol{i \in K}$ is selected according to (4). Then $\boldsymbol{\bar{y}_i = y_i}$, $\boldsymbol{\bar{\gamma}_i = \gamma_i = f(\bar{y}_i)}$, and, consequently, $\boldsymbol{f(y_i) = \gamma_i}$ is determined. But from (4) it follows that $\boldsymbol{f(y_i) \geq f^*}$. On the other hand, in view of (11) we have $\boldsymbol{f(y_i) \leq f^*}$. Thus, the equation $\boldsymbol{f(y_i) = f^*}$ is defined, and the theorem is proved.

Further, pay attention that there are some possibilities to update approximation sets $\boldsymbol{G_{i+1}}$ at iterations with numbers $\boldsymbol{i = i_k}$, and this is convenient from the practical viewpoint. These updates are performed on the basis of constructing sets $\boldsymbol{S_{i_k}}$. Namely, it will be shown below that for each $\boldsymbol{k \in K}$ during constructing $\{\boldsymbol{u_i}\}$, $\boldsymbol{i \in K}$, the number $\boldsymbol{i = i_k}$ will be fixed such that

$$\boldsymbol{||\bar{u}_i - u_i|| \leq \varepsilon_k}. \tag{12}$$

Then in view of (10) we can assign, for example, $\boldsymbol{S_{i_k} = G_{r_i}}$, where $\boldsymbol{0 \leq r_i \leq i_k - 1}$. In this case we can discard some cutting planes which are constructed to the iteration $\boldsymbol{i_k}$. On the other hand, the set $\boldsymbol{S_{i_k}}$ can be constructed on the basis of the last cutting planes, in particular, by getting active planes at the point $\boldsymbol{u_{i_k}}$.

Note that in view of (10) we can assign $\boldsymbol{S_{i_k} = G_{i_k}}$ for each $\boldsymbol{k \in K}$. Then the equality $\boldsymbol{S_i = G_i}$ is determined in (6) independently of conditions (5), (12), and updates of approximating sets do not occur.

Lemma 1. Suppose that the sequence $\{\boldsymbol{u_i}\}$, $\boldsymbol{i \in K}$, is constructed while $\boldsymbol{S_i}$ is defined in accordance with (8) for all $\boldsymbol{i \in K}$. Then

$$\boldsymbol{\lim_{i \in K} ||\bar{u}_i - u_i|| = 0}.$$

Proof. Assume the converse. Then there is a subsequence $\{\boldsymbol{u_i}\}$, $\boldsymbol{i \in K' \subset K}$, such that

$$||\bar{u}_i - u_i|| \geq \Delta > 0 \quad \forall i \in K'. \tag{13}$$

Select a convergence subsequence $\{u_i\}$, $i \in K'' \subset K'$, from the sequence $\{u_i\}$, $i \in K'$. Suppose i, $p_i \in K''$ such that $p_i > i$. From (6), (8) it follows that $G_{p_i} \subset G_i$. But $u_{p_i} \in G_{p_i}$, and, moreover, $b_i \in W(\bar{u}_i, G_{p_i})$. Consequently, we have $\langle b_i, u_{p_i} - \bar{u}_i \rangle \leq 0$. Since

$$\bar{u}_l = u_l + \alpha_l(v'' - u_l) \tag{14}$$

is defined for all $l \in K$, where $\alpha_l \in (0,1)$, then from (14) and the last inequality it follows that

$$\langle b_i, u_i - u_{p_i} \rangle \geq \alpha_i \langle b_i, u_i - v'' \rangle.$$

In view of $v'' \in int\ epi\ (f, R_n)$ it is not difficult to prove existence of the number $\sigma > 0$ such that

$$\langle b_l, v'' - u_i \rangle \leq -\sigma$$

for all $l \in K$. Therefore, we get $\langle b_i, u_i - u_{p_i} \rangle \geq \alpha_i \sigma$ or $||u_i - u_{p_i}|| \geq \alpha_i \sigma$. Further, since the sequence $\{u_i\}$, $i \in K''$, is convergence, then $\alpha_i \to 0$, $i \in K''$, is determined. Thus, from equality (14) for $l = i$ it follows that

$$||\bar{u}_i - u_i|| \to 0, \quad i \in K''.$$

This limit expression contradicts to (13). The lemma is proved.

Now on the basis of Lemma 1 lets prove that according to (9) the sequence $\{(x_k, \sigma_k)\}$ will be constructed with the sequence $\{u_i\}$.

Lemma 2. Suppose that the sequence $\{u_i\}$, $i \in K$, is constructed by the proposed method. Then there exist a number $i = i_k$ for each $k \in K$ such that (12) is defined.

Proof. Assume $k \in K$. Lets obtain inequality (12) for some $i = i_k$. Suppose the converse, i.e. inequality (5) is determined for all $i \in K$. Then sets S_i are given by (8) for all $i \in K$, and the statement of Lemma 1 contradicts to inequality (5).

Lemma 3. Let $\{x_k\}$, $k \in K' \subset K$, be a convergence subsequence of the sequence $\{x_k\}$, $k \in K$, and $\bar{x}$ be its limit point. Then the inclusion

$$\bar{x} \in D \tag{15}$$

is defined.

Proof. If the inclusion $x_k \in D$ is defined for infinitely many numbers $k \in K'$, then in view of closedness of the set D statement (15) is obviously obtained. That's why suppose that $x_k \notin D$ is given for all numbers $k \in K'$ such that $k \geq N \in K'$.

Note that

$$\bar{x}_k = x_k + \theta_k(v' - x_k), \quad k \in K', \quad k \geq N, \tag{16}$$

where $\theta_k \in (0,1)$. Lets choose numbers $l, p_l \in K'$ such that $p_l > l \geq N$. Since $M_{p_l} \subset M_l$, then $a_l \in W(\bar{x}_l, M_{p_l})$. But from (9), (13) it follows that $x_{p_l} \in M_{p_l}$, consequently, we have

$$\langle a_l, x_{p_l} - \bar{x}_l \rangle \leq 0,$$

and in view of (16) the inequality

$$\langle \boldsymbol{a_l}, \boldsymbol{x_l} - \boldsymbol{x_{p_l}} \rangle \geq \boldsymbol{\theta_l} \langle \boldsymbol{a_l}, \boldsymbol{x_l} - \boldsymbol{v}' \rangle \tag{17}$$

is defined. Since $\boldsymbol{x_k} \notin \boldsymbol{D}$, $\boldsymbol{k} \geq \boldsymbol{N}$, $\boldsymbol{k} \in \boldsymbol{K}'$ and $\boldsymbol{v}' \in \boldsymbol{int\, D}$, then there exist a number $\boldsymbol{\delta} > \boldsymbol{0}$ such that $\langle \boldsymbol{a_k}, \boldsymbol{v}' - \boldsymbol{x_k} \rangle \leq -\boldsymbol{\delta}$ for all $\boldsymbol{k} \geq \boldsymbol{N}$, $\boldsymbol{k} \in \boldsymbol{K}'$. Further, taking into account (17) and $||\boldsymbol{a_k}|| = \boldsymbol{1}$ the inequality

$$||\boldsymbol{x_l} - \boldsymbol{x_{p_l}}|| \geq \boldsymbol{\theta_k \delta}$$

is defined for any $\boldsymbol{l}$, $\boldsymbol{p_l} \in \boldsymbol{K}'$ such that $\boldsymbol{p_l} > \boldsymbol{l} \geq \boldsymbol{N}$. From this inequality and convergence of the sequence $\{\boldsymbol{x_k}\}$, $\boldsymbol{k} \in \boldsymbol{K}'$, it follows that $\boldsymbol{\theta_k} \to \boldsymbol{0}$, $\boldsymbol{k} \in \boldsymbol{K}'$. Then in accordance with (16) and boundness of the sequence $\{||\boldsymbol{v}' - \boldsymbol{x_k}||\}$, $\boldsymbol{k} \in \boldsymbol{K}'$, we get

$$||\overline{\boldsymbol{x}}_{\boldsymbol{k}} - \boldsymbol{x_k}|| \to \boldsymbol{0}, \quad \boldsymbol{k} \in \boldsymbol{K}'. \tag{18}$$

Now lets select a convergence subsequence $\{\overline{\boldsymbol{x}}_{\boldsymbol{k}}\}$, $\boldsymbol{k} \in \boldsymbol{K}'' \subset \boldsymbol{K}'$, from the sequence $\{\overline{\boldsymbol{x}}_{\boldsymbol{k}}\}$, $\boldsymbol{k} \in \boldsymbol{K}'$, and let $\widetilde{\boldsymbol{x}}$ be its limit point. Then in view of (18) we have $\widetilde{\boldsymbol{x}} = \overline{\boldsymbol{x}}$, and from closedness of the set $\boldsymbol{D}$ it follows that inclusion (15) is defined. The lemma is proved.

At last, lets obtain the theorem of convergence of the proposed method.

Theorem 2. The following expressions

$$\overline{\boldsymbol{x}} \in \boldsymbol{X}^*, \quad \overline{\boldsymbol{\sigma}} = \boldsymbol{f}^*$$

are defined for any limit point $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{\sigma}})$ of the sequence $\{(\boldsymbol{x_k}, \boldsymbol{\sigma_k})\}$, $\boldsymbol{k} \in \boldsymbol{K}$, which is constructed by the proposed method.

Proof. Let $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{\sigma}})$ be a limit point of the convergence subsequence $\{(\boldsymbol{x_k}, \boldsymbol{\sigma_k})\}$, $\boldsymbol{k} \in \boldsymbol{K}' \subset \boldsymbol{K}$, selected from the sequence $\{(\boldsymbol{x_k}, \boldsymbol{\sigma_k})\}$, $\boldsymbol{k} \in \boldsymbol{K}$. According to Lemma 3 the inclusion

$$\overline{\boldsymbol{x}} \in \boldsymbol{D}$$

is defined, i.e.

$$\boldsymbol{f}(\overline{\boldsymbol{x}}) \geq \boldsymbol{f}^*. \tag{19}$$

Choose a convergence subsequence $\{\overline{\boldsymbol{u}}_{\boldsymbol{i_k}}\}$, $\boldsymbol{k} \in \boldsymbol{K}'' \subset \boldsymbol{K}$ from the sequence $\{\overline{\boldsymbol{u}}_{\boldsymbol{i_k}}\}$, $\boldsymbol{k} \in \boldsymbol{K}'$, and let $\overline{\boldsymbol{u}}$ be its limit point. Since the set $\boldsymbol{epi}\ (\boldsymbol{f}, \boldsymbol{R_n})$ is closed, then

$$\overline{\boldsymbol{u}} \in \boldsymbol{epi}\ (\boldsymbol{f}, \boldsymbol{R_n}). \tag{20}$$

In accordance with condition (2) of constructing points $\boldsymbol{\varepsilon_k}$ and equations $\boldsymbol{u_{i_k}} = (\boldsymbol{x_k}, \boldsymbol{\sigma_k})$ which is obtained from inequalities $||\overline{\boldsymbol{u}}_{\boldsymbol{i_k}} - \boldsymbol{u_{i_k}}|| \leq \boldsymbol{\varepsilon_k}$, $\boldsymbol{k} \in \boldsymbol{K}''$, we have

$$\overline{\boldsymbol{u}} = (\overline{\boldsymbol{x}}, \overline{\boldsymbol{\sigma}}).$$

Then in view of (20) the inclusion $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{\sigma}}) \in \boldsymbol{epi}(\boldsymbol{f}, \boldsymbol{R_n})$ is determined, i.e.

$$\boldsymbol{f}(\overline{\boldsymbol{x}}) \leq \overline{\boldsymbol{\sigma}}. \tag{21}$$

But according to (11) $\boldsymbol{\sigma_k} \leq \boldsymbol{f}^*$, $\boldsymbol{k} \in \boldsymbol{K}''$, and $\overline{\boldsymbol{\sigma}} \leq \boldsymbol{f}^*$. Then from inequalities (19), (21) it follows that $\boldsymbol{f}(\overline{\boldsymbol{x}}) = \overline{\boldsymbol{\sigma}} = \boldsymbol{f}^*$. The theorem is proved.

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