

Solving of variational inequalities by reducing to the linear complementarity problem

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Abstract. We study the variational inequalities closely connected with the linear separation problem of the convex polyhedron in the Euclidean space. For solving of these inequalities, we apply the reduction to the linear complementarity problem. Such reduction allows one to solve the variational inequalities with the help of the Matlab software package.

1. Reduction to the projection problem

In this section, we demonstrate that the variational inequalities of our interest are reducible to the problem of projecting the origin of the Euclidean space onto the convex polyhedron.

In [1–4], we establish and study the close connection between the solutions of some variational inequalities and linear separation problem for two convex and closed sets A and B from the Euclidean space R^n . These inequalities consist in finding a vector $c \in R^n$ such that

$$\langle c, x - y \rangle \geq 0, \quad x \in A, \quad y \in B, \quad \Delta > 0; \quad (1)$$

$$\langle c, x - y \rangle > 0, \quad x \in A, \quad y \in B; \quad (2)$$

$$\langle c, x - y - c \rangle \geq 0, \quad x \in A, \quad y \in B; \quad (3)$$

$$\langle c, x - y \rangle \geq 0, \quad x \in A, \quad y \in B. \quad (4)$$

In regard the inequalities (3)-(4), of course, we are interested in nontrivial solutions.

The solution to the problem of the best strong linear separation of two sets A and B is also the solution for all above-enumerated variational inequalities. The best strong linear separation consists in finding the best (in the sense of separation thickness) support vector for the hyperplane strongly separating the sets A and B . This problem can be reduced to the problem of projecting the origin of R^n onto the Minkowski difference of these sets $A-B$. Let us note that the formulas for determining of the Minkowski difference for the different kinds of sets were proposed, for instance, in [2], [4]. In [2], there was proved that it holds

$$C := A - B = \text{conv}\{a_i - b_j\}, i = 1, \dots, m_1; j = 1, \dots, m_2,$$

$$\text{where } A = \text{conv}\{a_i\}, i = 1, \dots, m_1; B = \text{conv}\{b_j\}, j = 1, \dots, m_2,$$

i.e. $A-B$ is a convex polyhedron given as the convex hull of $m_1 \cdot m_2$ vectors $a_i - b_j$.

If for the projection of the origin onto C , it holds $P_C(\odot) = \odot$, then the inequalities (1)–(3) have no solutions. In this case, we can not state that (4) has no solutions too (there can exist the separator with null thickness). In this event, to solve (4) it is necessary to use, for example, maxi-min optimization problem presented in [3]. In [1], there was originally established the connection between the problem



of projecting [5] and the quadratic programming ones of a special type, associated with the cone of the strong support vectors for sets, onto which the projection is carried out.

Note that the terminology relating to the different kinds of the support vectors and the types of linear separation of sets, is studied in detail in [3]. In this paper, using the Kuhn-Tucker theorem, the quadratic programming problem (associated with the projection problem) is reduced to the classical linear complementarity problem. The solution of the linear complementarity problem can be obtained by using the function LCPSolve in MATLAB [5].

2. Connection of the variational inequalities with the linear complementarity problem

In the previous section, we noted that solving of (1)–(4) can be reduced to the projection problem. In this section, we study the reduction of the projection problem to the linear complementarity problem. As a result, we establish the connection of (1)–(4) with the linear complementarity problem.

Let us consider the following setting of the projection problem connected with the so-called inner representation of the convex polyhedron $C = \text{conv}\{z_i\}, i = 1, \dots, m$ [6].

$$\min \varphi(\alpha) = \left\| \sum_{i=1}^m \alpha_i z_i \right\|^2, \quad (5)$$

$$\sum_{i=1}^m \alpha_i = 1, \quad (6)$$

$$\alpha_i \geq 0, i = 1, \dots, m. \quad (7)$$

Further, the problem (5)–(7) can be easily rewritten in a vector-matrix form as follows

$$\min \varphi(\alpha) = \langle \alpha, D\alpha \rangle, \quad (8)$$

$$\langle e, \alpha \rangle = 1, \quad (9)$$

$$\alpha \geq 0, \quad (10)$$

Where D is $m \times m$ -dimensional matrix with elements

$$d_{ij} = z_i, z_j, i, j = 1, \dots, m; \quad e = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \quad e \in \mathbf{R}^m, \quad \alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix}.$$

Let α^* be an optimizer for the problem (8)-(10), then

$$P_C(\odot) = \sum_{i=1}^m \alpha_i^* z_i. \quad (11)$$

In [5], we prove that the matrix B is nonnegative definite. Consequently, the problem (8)-(10) is characterized as the convex quadratic programming one. We can rewrite the constraint (9) with the help of the two inequality constraints as follows.

$$\min \varphi(\alpha) = \langle \alpha, D\alpha \rangle,$$

$$\langle e, \alpha \rangle \geq 1,$$

$$\langle -e, \alpha \rangle \geq -1,$$

$$\alpha \geq 0.$$

Let us further construct the Lagrange function for the previous optimization problem:

$$L(\alpha, f) = \langle \alpha, D\alpha \rangle + \langle f, b - F\alpha \rangle, f \geq 0, f \in \mathbf{R}^2.$$

Where f is the vector of the Lagrange multipliers, F is the $2 \times m$ matrix

$$F = \begin{pmatrix} 1 & \dots & 1 \\ -1 & \dots & -1 \end{pmatrix}, b = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

It is not hard to check that it is fulfilled

$$L'_\alpha(\alpha, f) = 2D\alpha - F^T f,$$

$$L'_f(\alpha, f) = b - F\alpha.$$

It is well known that in the case of minimizing the convex function subject to the linear constraints, the optimality conditions for the problem (8)-(10) are determined as the saddle point conditions for the Lagrange function. Due to the Kuhn-Tucker theorem, it is necessary and sufficient to determine the vectors α, f such that

$$\begin{aligned} L'_\alpha(\alpha, f) &\geq 0, \\ L'_f(\alpha, f) &\geq 0, \\ \langle \alpha, L'_\alpha(\alpha, f) \rangle &= 0, \\ \langle f, L'_f(\alpha, f) \rangle &= 0, \\ f, \alpha &\geq 0. \end{aligned}$$

This system can be expressed as a linear complementarity problem [7] which consists in determining the vectors w and v such that

$$\begin{aligned} w &= Mv + q, \\ w &\geq 0, v \geq 0, \\ \langle w, v \rangle &= 0, \end{aligned}$$

where $M = \begin{pmatrix} 2D & | & -F^T \\ F & | & \theta \end{pmatrix}$ is the matrix with block structure from $R^{l \times l}$, $l = m + 2$;

w, v, q are $m+2$ -dimensional vectors; θ is a null matrix with dimension 2×2 ,

$$w = \begin{pmatrix} u \\ r \end{pmatrix}, v = \begin{pmatrix} \alpha \\ f \end{pmatrix}, q = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -b \end{pmatrix}, u = 2D\alpha - F^T f,$$

$$r = F\alpha - b; \alpha, f, u, r \geq 0; \langle u, \alpha \rangle = 0, \langle r, f \rangle = 0.$$

We further recall briefly the following cones defined in [3]:

$$\begin{aligned} W_C &= \{p \in R^n : \inf_{x \in C} \langle p, x \rangle \geq 0\}, \\ V_C &= \{h \in R^n : \inf_{x \in C} \langle h, x \rangle > 0\}, \quad Q_C = \{s \in R^n : \langle s, x \rangle > 0 \quad \forall x \in C\}, \\ \Omega_C &= \{g \in R^n : \langle g, x \rangle \geq \|g\|^2 \quad \forall x \in C\}, \\ E(\Omega_C) &= \{x \in R^n : x = \lambda g, \lambda \geq 0, g \in \Omega_C\}, \end{aligned}$$

where C is a nonempty subset of R^n , $W_C \setminus \{\odot\}$ is a cone of generalized support vectors of the set C . The notations V_C, Q_C correspond to the cones of generalized strong and strict support vectors of the set C , respectively. The main properties of the above-mentioned cones and their relationship with well-known other ones have been learned in [3]. The necessary and sufficient conditions for emptiness of the cones of generalized support vectors have been explored in [4]. We refer the reader to [3]-[4] for definitions, terminology, and details. Let us recall the important relationship of above-mentioned cones:

$$V_C \subseteq Q_C \subseteq W_C, \tag{12}$$

$$V_C = E(\Omega_C) \setminus \{\odot\}. \tag{13}$$

It is obvious that the variational inequalities (1),(2),(3), and (4) are associated with the cones V_C , Q_C , Ω_C , and W_C , respectively. In [3], for the case of the convex and compact set C it was proved that $V_C = Q_C$. If C is a convex and closed subset of R^n , then the operation of projection onto C is well defined. Due to Theorem 3.3 from [3], it holds $P_C(\odot) \in Bd(\Omega_C)$, where $Bd(\Omega_C)$ stands for the boundary of Ω_C . Due to Theorem 2.7 from [3], if the set C is convex and compact, then Ω_C is also a convex compact. Consequently, it is fulfilled $P_C(\odot) \in \Omega_C$, since $Bd(\Omega_C) \subseteq \Omega_C$. If we find the projection of the origin of R^n onto C and observe that $P_C(\odot) = \odot$. This means that $\odot \in C$. Owing to Lemma 3.6, we have $\Omega_C = \{\odot\}$. This implies that (3) is without non-zero solutions. Then from (13), by virtue of the fact $\Omega_C = \{\odot\}$, it follows that $V_C = \emptyset$, i.e. C is not strongly separable from the origin of R^n . Due to General theorem on strong separability from [3], the variational inequality (1) has no solutions as well. As for (4), we can not unambiguously asserts (under condition of $\odot \in C$) (4) has no non-zero solutions, too. For further analysis, we need to know whether or not the origin of R^n is the interior point of C . If it is, then (4) is without nontrivial solutions. In opposite case, i.e. when the origin belongs to the boundary of the set C [4], there can be found the non-zero solution for (4) using other tools (see [3]). If we obtain $P_C(\odot) \neq \odot$, then from above, due to (12)–(13), it holds $P_C(\odot) \in V_C, \Omega_C, W_C$. This means that $P_C(\odot)$ satisfies to (1)–(4).

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