On the Finite Element Approximations of Mixed Variational Inequalities of Filtration Theory

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Abstract. We construct the finite-element approximations for mixed variational inequalities with pseudomonotone operators and convex non-differentiable functionals in Sobolev spaces. Such variational inequalities arise in the mathematical description of the processes of an established filtration. The convergence of these approximations are investigated.

1. Introduction

Mixed variational inequalities with the monotone type operators [1, 2] arise in the mathematical description of the processes in the different areas of the science and technique, in particular, in the non linear filtration theory [3-7], in the problems of determining the equilibrium of soft shells [8-18], in the study of low-temperature plasma [19-24] etc. In this paper as a development of the research described in the paper [25] we construct and study the convergence of finite-element approximations for mixed variational inequalities of an established filtration with multivalued law. The existence of a subsequence of solutions of finite-element problems converging weakly to the solution of the original variational inequality is proved.

2. Problem statement

Let Ω be a bounded domain in R^m, $m \geq 1$, with a Lipschitz continuous boundary Γ and let $\xi \to g(\xi^2)\xi$ be the function that determines the filtration law. We assume that $g(\xi^2)\xi = g_0(\xi^2)\xi + g_1(\xi^2)\xi$, and

$$g_0(\xi^2) = 0, \quad \xi \geq \beta \tag{1}$$

($\beta \geq 0$ is a limiting gradient),

$$\text{function } \xi \to g_0(\xi^2)\xi \text{ is continuous,} \tag{2}$$

$$\text{function } \xi \to g_0(\xi^2)\xi \text{ strictly increase for } \xi \geq \beta \tag{3}$$

there exist $c_1 > 0$, $c_2 > 0$, $p > 1$, such that

$$c_1(\xi - \beta)^{p-1} \leq g_0(\xi^2)\xi \leq c_2(\xi - \beta)^{p-1} \text{ for } \xi \geq \beta, \tag{4}$$

$$g_1(\xi^2)\xi = \begin{cases} \vartheta, & \xi \geq \beta; \\ 0, & \xi < \beta. \end{cases}$$

Suppose that $\Gamma = \Gamma_0 \cup \Gamma_1$, $\operatorname{mes}\Gamma_1 > 0$, $V = \{\eta \in W_p^{(1)}(\Omega) : \eta(x) = 0,\, x \in \Gamma_0\}$, $M = \{\eta \in V : \eta(x) \geq 0,\, x \in \Gamma_1\}$ (Γ_1 is the semi-permeable part of the boundary), $A_0 : V \to V^* = W_{p^*}^{(-1)}(\Omega)$, $p^* = p/(p-1)$, is the operator generated by the form

$$\langle A_0 u, \eta \rangle = \int_\Omega g_0(|\nabla u|^2)(\nabla u, \nabla \eta)\,dx, \quad u, \eta \in V, \tag{5}$$

where $(\cdot,\cdot)$ and $|\cdot|$ are the inner product and norm in R^m.

We also define the functional $F_1 : V \to R^1$ by the formula

$$F_1(\eta) = \int_\Omega \int_0^{|\nabla \eta|} g_1(\xi^2)\xi\,d\xi\,dx = \vartheta \int_\Omega (|\nabla \eta| - \beta)^+\,dx, \quad a^+ = (a + |a|)/2.$$

By the solution to the filtration problem of an incompressible fluid with a multi valued filtration law with limiting gradient we mean the function $u \in M$, which is the solution of the variational inequality (see [26-28])

$$\langle A_0 u, \eta - u \rangle + F_1(\eta) - F_1(u) \geq \langle f, \eta - u \rangle \qquad \forall \eta \in M, \tag{6}$$

It is easy to verify that the set M is convex and closed. The functional F_1 is convex and Lipschitz continuous [29, 30]. If the conditions (1)–(4) hold then the operator A_0 is bounded, monotone, hemicontinuous [31] (and hence, pseudomonotone [31]) and coercive [31]. When the above conditions on the operator A_0 and the functional F_1 are imposed, problem (6) has at least one solution (see, e.g., [1, 31]).

3. Construction and investigation of finite element schemes for the filtration problems

In [25] have been considered the common case, when V is a reflexive Banach space with a uniformly convex dual space V^*, $\langle \cdot,\cdot \rangle$ is a duality relation between V and V^*, M is a closed convex set in V, the operator $A_0 : V \to V^*$ is pseudomonotone and coercive on M, i.e., for some $\hat{\eta} \in M$ the inequality $\langle A_0 \eta, \eta - \hat{\eta} \rangle \geq \rho(\|\eta\|_V)\|\eta\|_V$; $\lim\limits_{\xi \to +\infty} \rho(\xi) = +\infty$, is satisfied. Recall that the operator $A_0 : V \to V^*$ is called pseudomonotone [31] if it is bounded and the weak convergence of the sequence $\{u_k\}_{k=1}^{+\infty}$ in V to u^* and the inequality $\limsup\limits_{k \to +\infty} \langle A_0 u_k, u_k - u^* \rangle \leq 0$ implies that the following relation holds $\liminf\limits_{k \to +\infty} \langle A_0 u_k, u_k - \eta \rangle \geq \langle A_0 u^*, u^* - \eta \rangle$ for all η. Functional $F_1 : V \to R^1$ is a convex, continuous (but, in general, non-differentiable). Let $\{V_h\}_h$ be a family of spaces, where the parameter h tends to zero, such that $V_h \subset V$ for every h. Suppose that there given the linear operators $r_h : V \to V_h$ (restriction operators from V to V_h). We assume that the family $\{V_h\}_h$ approximates V, that is,

$$\lim_{h \to 0} \| r_h \eta - \eta \|_V = 0 \qquad \forall \eta \in V. \tag{7}$$

For each h let us consider a convex closed set $M_h \subset V_h$, approximating M, i.e., firstly, for any $\eta \in M$ an element $\eta_h \in M_h$ can be found, such that

$$\lim_{h \to 0} \| \eta_h - \eta \|_V = 0, \tag{8}$$

and, secondly,

$$\text{if } \eta_h \in M_h,\ \eta_h \to \eta \text{ weakly in } V \text{ for } h \to 0, \text{ then } \eta \in M. \tag{9}$$

Note that condition (9) is satisfied if $M_h \subset M$ for each h. Indeed, in this case, the family $\{\eta_h\}_h$ belongs to the weakly closed set M, hence, $\eta \in M$.

Let us associate problem (6) with a family of approximating problems of finding the elements $u_h \in M_h$, such that

$$\langle A_0 u_h, \eta_h - u_h \rangle + F_1(\eta_h) - F_1(u_h) \geq \langle f, \eta_h - u_h \rangle \quad \forall \eta_h \in M_h. \tag{10}$$

The following theorem holds [25].

Theorem 1. Let u_h be the solution of problem (10). Then there is a subsequence $\{h_k\}_{k=1}^{+\infty}$, $h_k \to 0$ for $k \to +\infty$, such that u_{h_k} converges weakly in V to a some solution u of problem (6) for $k \to +\infty$.

Moreover, any weak limit point u^* of the family $\{u_h\}_h$ is a solution of (6), and if $\{u_{h_k}\}_{k=1}^{+\infty}$ is subsequence converging weakly in V to u^* for $k \to +\infty$, then

$$\lim_{k\to+\infty} \langle A_0 u_{h_k} - A_0 u^*, u_{h_k} - u^* \rangle = 0. \tag{11}$$

In the construction of finite element schemes we will assume for simplicity that the boundary Γ consists of s-dimensional faces ($s \leq m-1$).

Let $\Im_h$ be a family (triangulation) of m-dimensional simplexes T, union of which coincides with Ω, and the intersection of two simplexes from $\Im_h$ may be only a s-dimensional face ($s \leq m-1$), $h = \sup_{T\in\Im_h} \rho_T$, $\sigma(h) = \sup_{T\in\Im_h} \rho_T / \rho'_T$ where ρ'_T is the diameter of the largest ball contained in T and ρ_T is the diameter of the smallest ball containing T. We assume that the triangulation $\Im_h$ is regular (see [32, 33]), i. e., $\sigma(h) \leq \hat{c}_1$, where the positive constant $\hat{c}_1$ is independent of h. We now define the finite-dimensional space V_h approximating V and associated with the triangulation $\Im_h$, as the space of continuous functions vanishing at Γ_0 and linear on each simplex $T \in \Im_h$. It is clear that $V_h \subset V$.

We denote by U_h the set of all vertices x_h of the simplexes $T \in \Im_h$, and by $\overset{\circ}{U}_h$ the set of all points of U_h, which belong to $\operatorname{int}\Omega$. It is obvious that $U_h \setminus \overset{\circ}{U}_h \subset \Gamma$. As a basis in V_h we will choose the functions $x \to \omega_h(x, x_h)$ defined on Ω for any $x_h \in \overset{\circ}{U}_h$, such that $\omega_h(\hat{x}_h, x_h) = \begin{cases} 1, & \hat{x}_h = x_h; \\ 0, & \hat{x}_h \neq x_h. \end{cases}$ Any function $u_h \in V_h$ in this case can be represented as a linear combination $u_h(x) = \sum_{x_h \in \overset{\circ}{U}_h} u_h(x_h)\,\omega_h(x, x_h)$.

Finally, we set $M_h = \{\eta_h \in V_h : \eta_h(x) = 0,\, x \in \Gamma_0\}$.

We associate problem (6) with an approximating problem, which consists in finding an element $u_h \in M_h$ which is the solution of the variational inequality (10) holds.

Let $(r_h\eta)(x) = \sum_{x_h \in \overset{\circ}{U}_h} \eta(x_h)\,\omega_h(x, x_h)$, $x \in \Omega$. The following results (from which it follows that the constructed mesh schemes satisfy the conditions (7)–(9) hold.

Lemma 1. Let $\eta \in V$, then $r_h \eta \in V_h$, $\lim_{h\to 0} \| r_h\eta - \eta \|_V = 0$.

Lemma 2. Let $\eta \in M$, then $r_h \eta \in M_h$, $\lim_{h\to 0} \| r_h\eta - \eta \|_V = 0$, and the property {9} is satisfied.

The validity of Lemma 1 follows directly from the results of [34], and the validity of Lemma 2 follows from the definition of the sets M, M_h and Lemma 1.

We also need the following result [35].

Lemma 3. Let the conditions (1)–(4) hold, the sequence $\{u^{(n)}\}_{n=1}^{+\infty} \subset V$ converges weakly in V for $n \to +\infty$ to u and the operator $A_0 : V \to V^*$ generated by the (5) satisfies the relation

$\lim_{n\to+\infty}\left\langle A_0 u^{(n)} - A_0 u, u^{(n)} - u^{*}\right\rangle = 0$. Then, $\lim_{n\to+\infty}\int_{\Omega_{+u}} |\nabla(u^{(n)} - u)|^p\,dx = 0$, where $\Omega_{+u} = \{x\in\Omega : |\nabla u(x)| > \beta\}$ is the flow domain.

We have the following

Theorem 2. Problem (10) has at least one solution. The family of the solutions to (10) is uniformly bounded in h. Any weak limit point u of the family $\{u_h\}_h$ is a solution of {6}, and if $\{u_{h_k}\}_{k=1}^{+\infty}$ is a subsequence which converges weakly for $k\to+\infty$ in V to u, then

$$\lim_{k\to+\infty}\left\| u_{h_k} - u \right\|_{L_p(\Omega)} = 0, \tag{12}$$

$$\lim_{k\to+\infty}\int_{\Omega_{+u}} |\nabla(u_{h_k} - u)|^p\,dx = 0. \tag{13}$$

Proof. The existence of the solutions to (10) and the fact that any weak limit point u of the family $\{u_h\}_h$ is a solution of (10) follow from Theorem 1. Relation (12) follows from the compactness of the embedding V in $L_p(\Omega)$ and relation (13) follows from (11) and Lemma 3.

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References

[1] Ekeland I and Temam R 1976 *Convex Analysis and Variational Problems* (Amsterdam: North-Holland)
[2] Glowinski R Lions J L and Tremolieres R 1976 *Analyse numérique des inéquations variationnelles* (Paris: Dunod)
[3] Badriev I B and Karchevskii M M 1994 Convergence of an iterative process in a Banach space *Journal of Mathematical Sciences* **71 (6)** 2727-35. DOI: 10.1007/bf02110578 15
[4] Badriev I B and Karchevskii M M 1989 Convergence of the iterative Uzawa method for the solution of the stationary problem of seepage theory with a limit gradient *Journal of Soviet Mathematics* **45 (4)** 1302-9 DOI: 10.1007/BF01097083
[5] Badriev I B and Nechaeva L A 2013 Mathematical simulation of steady filtration with multivalued law *PNRPU Mechanics Bulletin* (3) 37–65
[6] Lapin A V 2010 Preconditioned Uzawa-Type Methods for Finite-Dimensional Constrained Saddle Point problems *Lobachevskii Journal of Mathematics* **31 (4)** 309-322. DOI 10.1134/s1995080210040013
[7] Shangaraeva A I and Shevchenko D V 2015 Speed up of the oil saturation numerical algorithm for the plane-parallel filtration *Applied Mathematical Sciences* **9 (150)** 7467-74 DOI 10.12988/ams.2015.510683
[8] Badriev I and Banderov V 2014 Numerical method for solving variation problems in mathematical physics *Applied Mechanics and Materials* **668-669** 1094-97. DOI: 10.4028/www.scientific.net/AMM.668-669.1094
[9] Abdrakhmanova A I and Sultanov L U 2016 Numerical modelling of deformation of hyperelastic incompressible solids *Materials Physics and Mechanics* **26 (1)** 30-2
[10] Badriev I B, Banderov V V, Garipova G Z, Makarov M V and Shagidullin R R 2015 On the solvability of geometrically nonlinear problem of sandwich plate theory *Applied Mathematical Sciences* **9 (82)** 4095-102. DOI: 10.12988/ams.2015.54358
[11] Badriev I B, Banderov V V, Makarov M V and Paimushin V N 2015 Determination of stress-strain state of geometrically nonlinear sandwich plate *Applied Mathematical Sciences* **9 (78)**

3887–95. DOI: 10.12988/ams.2015.54354

[12] Badriev I B, Banderov V V and Zadvornov O A 2013 On the solving of equilibrium problem for the soft network shell with a load concentrated at the point *PNRPU Mechanics Bulletin* **(3)** 17-35

[13] Badriev I B and Shagidullin R R 1995 A study of the convergence of a recursive process for solving a stationary problem of the theory of soft shells *Journal of Mathematical Sciences* **73 (5)** 519-25 DOI :10.1007/bf02367668

[14] Berezhnoi D V, Paimushin V N and Shalashilin V I 2010 Studies of quality of geometrically nonlinear elasticity theory for small strains and arbitrary displacements *Mechanics of Solids* **44 (6)** 837-51. DOI 10.3103/S0025654409060041

[15] Davydov R L and Sultanov L U 2014 Mathematical modeling of large elastic-plastic deformations *Applied Mathematical Sciences* **60 (8)** 2991-6 DOI 10.12988/ams.2014.44272

[16] Sagdatullin M and Berezhnoi D 2014 Statement of the problem of numerical modelling of finite deformations *Applied Mathematical Sciences* **8 (33-36)** 1731-8 DOI 10.12988/ams.2014.4283

[17] Sultanov L U 2014 Mathematical modeling of deformations of hyperelastic solids *Applied Mathematical Sciences* **8 (143)** 7117-24 DOI 10.12988/ams.2014.49704

[18] Sultanov L U and Davydov R L 2014 Mathematical modeling of large elastic-plastic deformations *Applied Mathematical Sciences* **8** (60) 2991-6 DOI 10.12988/ams.2014.44272

[19] Chebakova V Ju, Zheltukhin V S and Dubrovin V T 2016 Problem of radio-frequency discharge at atmospheric pressure in local statement *Applied Mathematical Sciences* **10 (21)** 1013-22

[20] Zheltukhin V S, Chebakova V J, Abaidullin R N and Kadyrov R F 2015 On solving of some nonlinear problems of plazma physics *Applied Mathematical Sciences* **9** (88) 4351-9

[21] ZheltuhinV S, Solovyev P S and Chebakova V Y 2015 Boundary conditions for electron balance equation in the stationary High-Frequency Induction discharges *Research Journal of Applied Sciences* **10 (10)** 658-62

[22] Dubrovin V T, Gabbasov F G and Chebakova V J 2016 Multidimensional central limit theorem for sums of functions of the trajectories of endomorphisms *Lobachevskii Journal of Mathematics* **37 (4)** 409-17

[23] Dubrovin V T, Chebakova V Ju and Zheltukhin V S 2016 Radio-Frequency Discharge at Low Pressure: a Non-Local Problem Statement Approach *Procedia Engineering* **150** 1041–5

[24] Zheltukhin V S, Solov'ev S I, Solov'ev P S and Chebakova V Yu 2014 Computation of the minimum eigenvalue for a nonlinear Sturm–Liouville problem *Lobachevskii J. Math.***35 (4)** 416–26

[25] Badriev I B, Banderov V V, Gnedenkova V L, Kalacheva N V, Korablev A I and Tagirov R R 2015 On the finite dimensional approximations of some mixed variational inequalities *Applied Mathematical Sciences* **9 (114)** 5697-705 DOI. 10.12988/ ams.2015.57480

[26] Badriev I B 2013 On the Solving of Variational Inequalities of Stationary Problems of Two-Phase Flow in Porous Media *Applied Mechanics and Materials* **392** 183-7 DOI: 10.4028/www.scientific.net/AMM.392.183

[27] Badriev I B 1983 Difference schemes for nonlinear problems of filtration theory with a discontinuous law *Soviet Mathematics* **27** (5) 1–11

[28] Badriev I B and Fanyuk B Y 2012 Iterative methods for solving seepage problems in multilayer beds in the presence of a point source *Lobachevskii Journal of Mathematics* **33 (4)** 386-99 DOI 10.1134/S1995080212040026

[29] Badriev I B, Zadvornov O A and Lyashko A D 2004 A study of variable step iterative methods for variational inequalities of the second kind *Differential Equations* **40 (7)** 971-83

[30] Badriev I B, Zadvornov O A and Saddek A M 2001 Convergence Analysis of Iterative Methods for Some Variational Inequalities with Pseudomonotone Operators *Differential Equations* **37 (7),** 934-42 DOI 10.1023/a:1011901503460

[31] Lions J L 1969 *Quelque problèmes méthodes de résolution des problèmes aux limites*

IOP Conf. Series: Materials Science and Engineering **158** (2016) 012012 doi:10.1088/1757-899X/158/1/012012

nonlinéaires (Paris: Dunod)
[32] Ciarlet P 1978 *The Finite Element Method For Elliptic Problems* (Amsterdam: North-Holland)
[33] Temam R 1977 *Navier-Stokes Equations* (Amsterdam: North-Holland)
[34] Ladyzhenskaya O A 1985 *The Boundary Value Problems of Mathematical Physics* (New York: Springer-Verlag)
[35] Badriev I B and Zadvornov O A 1966 Convergence of an iterative process for equations with degenerate operators *Differential Equations* **32 (7)** 902-5