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Udzawa-type iterative method with parareal preconditioner for a parabolic optimal control problem

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Abstract. The article deals with the optimal control problem with the parabolic equation as state problem. There are point-wise constraints on the state and control functions. The objective functional involves the observation given in the domain at each moment. The conditions for convergence Udzawa's type iterative method are given. The parareal method to inverse preconditioner is given. The results of calculations are presented.

1. Introduction

The numerical solution of parabolic control problems requires multiple solving the direct and adjoint equations (cf., e.g., [1]). The corresponding simulations can be too expensive and lead to the difficulty in obtaining "real-time solutions". To overcome this difficulty the parallel algorithms for multiprocessor workstations are usually constructed. A common way to construct a parallel algorithm is based on the decomposition of a spatial domain in elliptic problems as well as in evolutionary problems. Another approach consists in splitting the global problem into a series of independent evolutionary problems on smaller time intervals. It is so-called parareal algorithm, proposed in [2], [3] and widely used for solving different scientific and applied problems (cf., e.g., [4]-[9] and the bibliographies therein).

In this article we use the results of [10], [11] to construct a preconditioned Udzawa-type iterative method for mesh approximation of a state constrained parabolic optimal control problem. For the implementation of this method we propose a parareal algorithm of inverting the preconditioning matrix.

2. Initial problem, mesh approximation

Let $\Omega = (0,1)^n$, $n \geq 1$, with the boundary $\partial\Omega$, $Q_T = \Omega \times (0,T]$ and $\Sigma_T = \partial\Omega \times (0,T]$. A parabolic initial-boundary value problem

$$y_t - \Delta y = u \text{ in } Q_T, \quad y = 0 \text{ on } \Sigma_T, \quad y = 0 \text{ for } t = 0,\ x \in \Omega \tag{1}$$

is a state problem. For $u \in L_2(Q_T)$ there exists a unique solution of problem (1) from the space

$W(Q_T) = \{y \in L_2(0,T;H_0^1(\Omega)),\ y_t \in L_2(Q_T)\}$ ([12], p.370). We define the sets of constraints for control u and state y:

$$U_{ad} = \{u \in L_2(Q_T) : |u(x,t)| \leq u_{\max}\}, \quad Y_{ad} = \{y \in W(Q_T) : y_{\min} \leq y(x,t) \leq y_{\max}\} \quad \forall (x,t) \in Q_T.$$

Above u_{max}>0 and $-\infty \le y_{min} < 0 < y_{max} \le +\infty$. Let objective function be defined by the equality

$$J(y,u) = \frac{1}{2}\int_{Q_T} (y(x,t) - y_d(x,t))^2\, dxdt + \frac{1}{2}\int_{Q_T} u^2(x,t)\, dxdt \tag{2}$$

with a given observation function $y_d \in L_2(Q_T)$. Optimal control problem reads as follows:

$$\begin{gathered}\min_{(y,u)\in K} J(y,u),\\ K = \{(y,u) \in Y_{ad} \times U_{ad} : equation(1)\ holds\}.\end{gathered} \tag{3}$$

It is a minimization problem of quadratic functional $J(y, u)$ on a closed, bounded and convex set $K \neq \varnothing$, so, it has a unique solution (cf. [1]).

Below we suppose for the simplicity that the function y_d is continuous. We construct a finite difference approximation of problem (3) on the uniform grid $\omega_x \times \omega_\tau$ in $\overline{Q}_T$, where ω_x is a grid with a step h while $\omega_\tau = \{t_j = j\tau,\ j = 0,1,\ldots,M_\tau;\ M_\tau \tau = T\}$. Let V_h be the space of mesh functions defined on the grid ω_x vanishing in the boundary nodes $\partial\omega_x$ and $y_j = y(x,t_j) \in V_h$ be a mesh function on a time level $t_j = j\tau \in \omega_\tau$. Later we use the same notations both for mesh functions and the vectors of their nodal values. By N_x we denote the dimension of V_h and by $\|\cdot\|_x$ the Euclidian norm of the vectors of nodal values in this space.

Let A be well-known matrix of mesh Laplace operator on the grid ω_x with homogeneous Dirichlet boundary conditions. Approximate state problem (1) using backward Euler finite difference method:

$$\frac{y^j - y^{j-1}}{\tau} + Ay^j = u^j,\ j = 1,2,\ldots,M_\tau, \quad y^0 = 0. \tag{4}$$

The approximation of the objective function (2) on the grid $\omega_x \times \omega_\tau$ is the sum $\frac{\tau h^n}{2}\left(\sum_{j=1}^{M_\tau}\|y^j - y_d^j\|_x^2 + \sum_{j=1}^{M_\tau}\|u^j\|_x^2\right)$. We scaled it to derive the following mesh objective function:

$$J(y,u) = \frac{1}{2}\sum_{j=1}^{M_\tau}\|y^j - y_d^j\|_x^2 + \frac{1}{2}\sum_{j=1}^{M_\tau}\|u^j\|_x^2. \tag{5}$$

The sets of the constraints for the mesh control and state functions we define as follows

$$\begin{aligned}U_{ad}^h &= \{(u^1,\ldots,u^{M_\tau}) : u^j \in V_h, |u^j| \le u_{\max}\},\\ Y_{ad}^h &= \{(y^1,\ldots,y^{M_\tau}) : y^j \in V_h, y_{\min} \le y^j \le y_{\max}\}.\end{aligned}$$

Now, mesh optimal control problem reads as follows:

$$\begin{gathered}\min_{(y,u)\in K_h} J(y,u),\\ K_h = \{(y,u) \in Y_{ad}^h \times U_{ad}^h : equation(4)\ holds\}.\end{gathered} \tag{6}$$

Problem (6) is a minimization problem of a quadratic function on a compact set, so, it has a unique solution.

Later we consider problem (6) with very big number of time levels M_τ and use a parareal algorithm for solving (4). Namely, let $\Delta t >> \tau$ be a new time step and $\omega_{\Delta t} = \{t_j = j\Delta t,\ j = 0,1,\ldots,M_{\Delta t};\ M_{\Delta t}\Delta t = T\}$. Further for the simplicity we suppose that $\Delta t = m\tau$ with $m >> 1$.

Following the ideas and terminology of the article [13], we define as a coarse propagator backward Euler finite difference scheme on the coarse grid $\omega_{\Delta t}$, while a fine propagator - backward Euler finite difference scheme on the fine grid ω_τ.

3. Iterative method for the mesh problem

Let $N = N_x M_\tau$ be the dimension of the mesh functions of the variables x and t, $E \in R^{N\times N}$ be unit matrix and $L \in R^{N\times N}$ be defined by the equality

$$(Ly)^j = \left\{ \frac{y^1}{\tau}\ for\ j=1;\quad \frac{y^j - y^{j-1}}{\tau} + Ay^j\ for\ j = 2,\ldots,M_\tau \right\}.$$

Let also ψ and φ be the indicator functions of the sets Y_{ad}^h and U_{ad}^h, respectively, while $\partial\psi$ and $\partial\varphi$ be their subdifferentials. Mesh optimal control problem (5) can be rewritten as

$$\min_{Ly=u}\{J(y,u) + \psi(y) + \varphi(u)\}.$$

Lagrange function for this problem is defined by the equality

$$\Im(y,u,\lambda) = J(y,u) + \psi(y) + \varphi(u) + (\lambda, Ly - u),$$

where $(\cdot,\cdot)$ is Euclidian inner product in R^N.

Due to theory of saddle point problems ([14], p.169) a saddle point of this Lagrange function satisfies the following system:

$$\begin{pmatrix} E & 0 & L^T \\ 0 & E & -E \\ L & -E & 0 \end{pmatrix}\begin{pmatrix} y \\ u \\ \lambda \end{pmatrix} + \begin{pmatrix} \partial\psi(y) \\ \partial\varphi(u) \\ 0 \end{pmatrix} \ni \begin{pmatrix} y_d \\ 0 \\ 0 \end{pmatrix}. \tag{7}$$

With the notations $z = (y, u)^T$, $f = (y_d, 0, 0)^T$, $\Psi(z) = (\psi(y), \varphi(u))^T$ and

$$A = diag(E\quad E),\quad B = (L\quad -E)$$

problem (7) reads as

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}\begin{pmatrix} z \\ \lambda \end{pmatrix} + \begin{pmatrix} \partial\psi(z) \\ 0 \end{pmatrix} \ni \begin{pmatrix} f \\ 0 \end{pmatrix}. \tag{8}$$

Using the results of [15] we can prove the existence of a solution to (6) and convergence of a preconditioned Udzawa method

$$\begin{aligned} &Az^{k+1} + \partial\psi(z^{k+1}) \ni f, \\ &\frac{1}{\rho}D(\lambda^{k+1} - \lambda^k) + Bz^{k+1} = 0,\quad \rho > 0. \end{aligned} \tag{9}$$

with preconditioning matrix $D = D^T > 0$ and iterative parameter ρ which satisfy the inequality

$$D > \frac{\rho}{2} BA^{-1}B^T = \frac{\rho}{2}(LL^T + E). \tag{10}$$

In [16] the convergence result has been proved for method (9) with the preconditioner $D = LL^T$ and iterative parameter $\rho \in (0,1)$. The detailed form of the corresponding iterative method reads as follows:

$$\begin{aligned} &u^{k+1} + \partial\varphi(u^{k+1}) \ni \lambda^k, \\ &y^{k+1} + \partial\psi(y^{k+1}) \ni y_d - L^T\lambda^k, \\ &LL^T \frac{\lambda^{k+1} - \lambda^k}{\rho} = Ly^{k+1} - u^{k+1}. \end{aligned} \tag{11}$$

Implementation of every step in (11) consists of solving the inclusions to find u^{k+1} and y^{k+1}, and solving a system of linear equations with the matrix LL^T. Since the operators $\partial\varphi$ and $\partial\psi$ have diagonal forms, vectors u^{k+1} and y^{k+1} can be found component-wise by the explicit formulae. So, the most time consuming part of the algorithm is solution of the system with the matrix LL^T. In the following section we describe a parareal algorithm of inverting matrix LL^T.

4. Parareal preconditioner

To find the solution of the third equation in system (10) we have to invert the matrix

$$L = \begin{pmatrix} \tau^{-1}E & 0 & \dots & 0 & 0 \\ -\tau^{-1}E & -\tau^{-1}E + A & \dots & 0 & 0 \\ 0 & -\tau^{-1}E & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & -\tau^{-1}E & -\tau^{-1}E + A \end{pmatrix}$$

and its transposed L^T. Denote by $g = Ly^{k+1} - u^{k+1}$. Then we get from (11) the equation $Lv = g$, where $v = L^T \frac{\lambda^{k+1} - \lambda^k}{\rho}$.

We use the following variant of parareal method for finding a solution v of the equation $Lv = g$.

Let $\tilde{v}^k$ be calculated using backward Euler finite difference scheme on a coarse grid $\omega_{\Delta t} = \{t_j = j\Delta t,\ j = 0,1,\dots,M_{\Delta t};\ M_{\Delta t}\Delta t = T\}$ with $\Delta t = m\tau, \quad m >> 1$.

Then at each time subinterval $[j\Delta t, (j+1)\Delta t],\ j = 0,1,\dots,M_{\Delta t} - 1$ we use the values of $\tilde{v}_j^k = \tilde{v}^k(t_j)$ as initial values to find a solution v^k in the points $t_0 = j\Delta t, t_1 = j\Delta t + \tau, \dots, t_m = (j+1)\Delta t,\ j = 0,1,\dots,m$ of the fine grid:

$$\frac{v^k(t_{j+1}) - v^k(t_j)}{\tau} + Av^k(t_{j+1}) = g(t_{j+1}), \quad v^k(t_0) = \tilde{v}^k(j\Delta t). \tag{12}$$

After that we define a correction vector $\delta^k = v^k - \tilde{v}^k$ in the points of coarse grid and find a new value $\tilde{v}^k$ via following sequential procedure for $t_j \in \omega_{\Delta t}$:

$$\frac{w-\tilde{v}^k(t_j)}{\Delta t}+Aw=g(t_{j+1}),\quad \tilde{v}^k(t_{j+1})=w+\tilde{\delta}^k(t_{j+1}).$$

The main feature of (12) is the possibility of its parallel implementation. Namely, we can use $M_{\Delta t}$ processors for searching v^k in $M_{\Delta t}$ subintervals in parallel manner.

Denoting $w=\dfrac{\lambda^{k+1}-\lambda^k}{\rho}$ we get $L^T z=v$. The implementation of this equation differs from the algorithm above in that all calculations start from the last time-point *t*=*T* and continue backward for $j=M_{\Delta t},M_{\Delta t}-1,\dots,1$. Finally, $\lambda^{k+1}=\lambda^k+\rho z$.

The stability of the parareal algorithm is proved in the article [17]. In addition, the results of [18] show us that suitably damping coarse schemes ensure unconditional stability of the parareal algorithm.

5. Numerical experiments

The parareal algorithm is realized with the following data. For simplicity we choose one-dimensional case, when $n = 1$, and there are no restrictions on functions y and u. The final time moment is $T = 1$, mesh size $h = 0.05$. Time step of the fine grid $t = h$, the step of the coarse grid $\tau = h^2$. The iterative parameter $\rho = 0.005$, the observation function $y_d(x, t) = e^t \sin(\pi x)$. The calculation stopping criterion is number of iterations in Udzawa method.

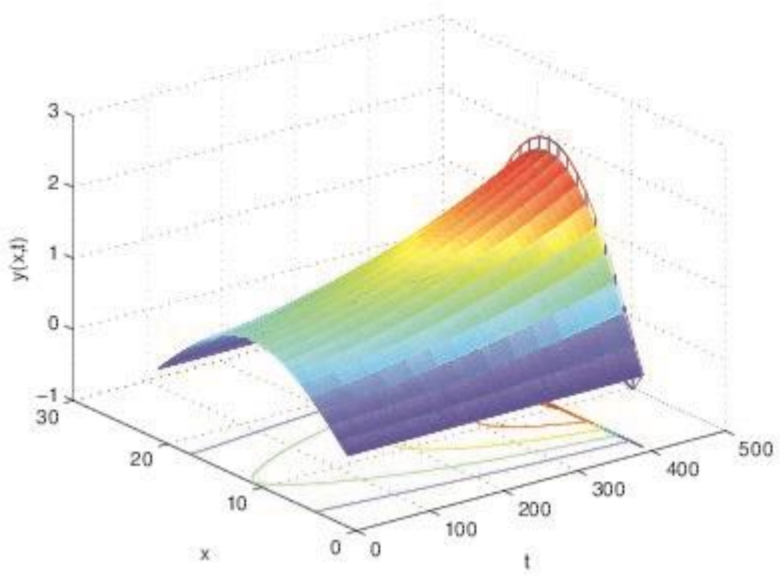

Figure 1. Shape of the state function *y*.

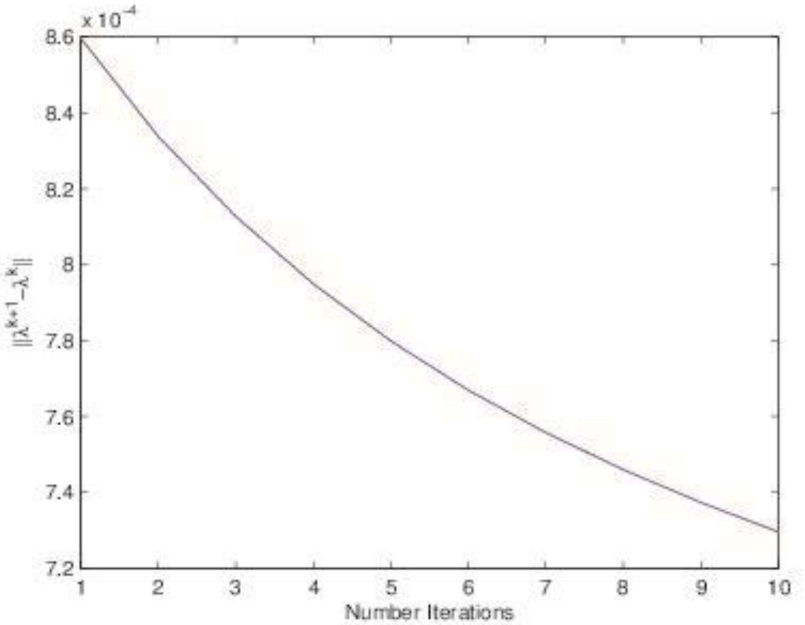

Figure 2. Residual $\left\|\lambda^{k+1}-\lambda^{k}\right\|_{L_2(Q_T)}$.

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