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Numerical investigation of large strains of hyperelastic solids

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Abstract. The work is concerned with numerical investigation of large deformations of hyperelastic solids. The kinematics of continua is described in terms of the deformation gradient tensor and left Cauchy–Green tensor. The stress state is described by the Cauchy stress tensor. An incremental method is used to solve the nonlinear problems. The linearized physical equations for the Cauchy stresses for Mooney–Rivlin material are given.

1. Introduction

Different rubber-like materials, elastic polymers, foams and even biological tissues [8, 9] can be described by nonlinear rheological models. According the mechanics of solids, hyperelastic materials are nonlinear. So while studying their deformations we must consider geometrical nonlinearity in terms of large deformations. There are plenty of different approaches described in literature [1–7].

2. Kinematics. Constitutive equations

Basic tensor of the kinematics of continua is deformation gradient tensor (F). The following tensors are used to describe strain and strain rate [10, 15–25]:

$(B)=(F)\cdot(F)^T$ – the left Cauchy–Green tensor (Finger deformation tensor),

$(h)=(\dot{F})\cdot(F^{-1})$ – the velocity gradient tensor,

$(d)=\frac{1}{2}\left[(h)+(h)^T\right]$ – the rate of deformation tensor.

Stress state is described by Cauchy tensor of true stresses (Σ) defined in the actual state. As arguments of strain energy function we take Finger tensor [11, 15–25]:

$$W=W(B). \tag{1}$$

To divide volumetric and shearing strains non-volumetric strain measures are considered:

$$(\hat{B})=J^{-2/3}(B),$$

here $J=dV/dV_0$ is the ratio of the deformed elastic volume over the reference (undeformed) volume of materials.

Then (1) will become:

$$W=W_0(J)+W'(I_{1\hat{B}},I_{2\hat{B}}).$$

Cauchy stress tensor in this case expresses as:

$$(\Sigma) = \frac{2}{J}(F)\cdot(F)^T \cdot \left(\frac{\partial W}{\partial B}\right) = \frac{2}{J}(B)\cdot\left(\frac{\partial W}{\partial B}\right).$$

Cauchy stress rate:

$$(\dot{\Sigma}) = (\Lambda_\Sigma)\cdot\cdot(d) + (h)\cdot(\Sigma) + (\Sigma)\cdot(h)^T - (\Sigma)I_{1d},$$

where $(\Lambda_\Sigma) = \frac{4}{J}(B)\cdot\left(\frac{\partial^2 W}{\partial B \partial B}\right)\cdot(B)$ [12, 15–25].

3. Calculation algorithm

To obtain numerical solution of the task Upgrade Lagrange formulation is used. As resolving expression principle of virtual work, written for the step $(k+1)$ is taken [12, 17, 23]:

$$\iiint\limits_{V_{k+1}} \left({}^{k+1}\Sigma\right)\cdot\cdot\left(\delta\, {}^{k+1}d\right) dV = \iiint\limits_{V_{k+1}} {}^{k+1}\vec{f}\cdot\delta\vec{\upsilon}\, dV + \iint\limits_{S^{\sigma}_{k+1}} {}^{k+1}\vec{t}_n \cdot \delta\vec{\upsilon}\, dS, \tag{2}$$

here V_{k+1} – actual volume, S^{σ}_{k+1} – loaded surface, $\vec{f}, \vec{t}_n$ – body and surface force vectors.

After linearization and rewriting Cauchy stress tensor in increments:

$$\left({}^{k+1}\Sigma\right) = \left({}^{k}\Sigma\right) + \left(\Delta^{k}\Sigma\right) \tag{3}$$

in expression (2) displacement vector for current configuration $\Delta^k \vec{u} = \Delta^k x_i \vec{e}_i$ could be obtained. This vector defines next step configuration:

$${}^{k+1}\vec{R} = {}^{k}\vec{R} + \Delta^k \vec{u}. \tag{4}$$

And stress tensor in this case will be:

$$\left({}^{k+1}\Sigma\right) = \frac{2}{J}\left({}^{k+1}B\right)\cdot\left(\frac{\partial W}{\partial\, {}^{k+1}B}\right). \tag{5}$$

4. Numerical example

As an example obtaining physical equations for material described by the Mooney–Rivlin elastic strains potential function [14]:

$$W = U_1(I_{1B} - 3) + U_2(I_{2B} - 3) + \frac{K}{2}(J-1)^2,$$

here $W_0(J) = \frac{K}{2}(J-1)^2$, $W'(I_{1B}, I_{2B}) = U_1(I_{1B} - 3) + U_2(I_{2B} - 3)$, U_1, U_2 – material constants.

For this material Cauchy stress tensor expresses as:

$$(\Sigma) = (\Sigma_0) + (\Sigma'),$$

where $(\Sigma') = \frac{2}{J}(B)\cdot\left(\frac{\partial W'}{\partial B}\right)$ – non-volumetric part, $(\Sigma_0) = \frac{2}{J}(B)\cdot\left(\frac{\partial W_0}{\partial B}\right)$ – volumetric part.

So:

$$(\Sigma) = 2U_1 J^{-\frac{5}{3}}\left[(B) - \frac{1}{3}I_{1B}(I)\right] + 2U_2 J^{-\frac{7}{3}}\left[I_{1B}(B) - \frac{1}{3}I_{1B}^2(I) - \frac{2}{3}(B^2)\right] + K(J-1)(I). \tag{6}$$

Linearizing (6), expression for Cauchy stress rate could be obtained:

$$(\Delta\Sigma) = (\Lambda_\Sigma)\cdot\cdot(d) + (h)\cdot(\Sigma) + (\Sigma)\cdot(h)^T - (\Sigma)I_{1d},$$

where $\left(\Lambda_{\Sigma}\right)=\left(\Lambda_{\Sigma'}\right)+\left(\Lambda_{\sigma 0}\right)$, $\Lambda_{\Sigma'}=\frac{4}{J}(B)\cdot\left(\frac{\partial^2 W'}{\partial B \partial B}\right)$ – non-volumetric part, $\Lambda_{\sigma 0}=\frac{4}{J}(B)\cdot\left(\frac{\partial^2 W_0}{\partial B \partial B}\right)$ – volumetric part.

It gives

$$\Lambda_{\Sigma}=\frac{4}{3}J^{-\frac{5}{3}}U_1\left\{\frac{1}{3}I_{1B}(I)(I)-(I)(B)-(B)(I)+I_{1B}(C_{II})\right\}+$$
$$+\frac{4}{3}J^{-\frac{7}{3}}U_2\left\{\frac{2}{3}I_{1B}^2(I)(I)-2I_{1B}(I)(B)+\frac{4}{3}(I)(B^2)+3(B)(B)-2I_{1B}(B)(I)-\right.$$
$$\left.-I_{1B}^2(C_{II})-2(B)\cdot(C_{II})\cdot(B)\right\}+K\left\{(J-1)(I)(I)+J(I)(I)+2(J-1)(C_{II})\right\}$$

So system (2)–(5) can be implemented to define stress state of hyperelastic materials which can be discretized by finite element method to obtain numerical solutions.

To prove efficiency of described algorithm it was implemented to some test problems. Here is one of them:

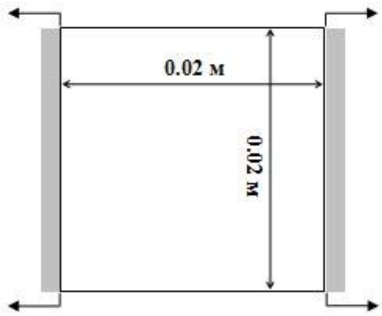

Figure 1. Plain deformation of square strip.

Plain deformation of square strip [13, 14, 17, 23] $0.02\times0.02\ \mathrm{m}^2$, material properties set as $U_1=0.36M\mathrm{Pa}$, $U_2=0{,}22MPa$, $K=2000\,\mathrm{MPa}$; displacements defined on the vertical sides of the strip as shown on figure 1 taking into account symmetry of the problem only 0.25 of strip was considered. On figure 2 vertical displacements field of the 0.25 part of the strip is shown.

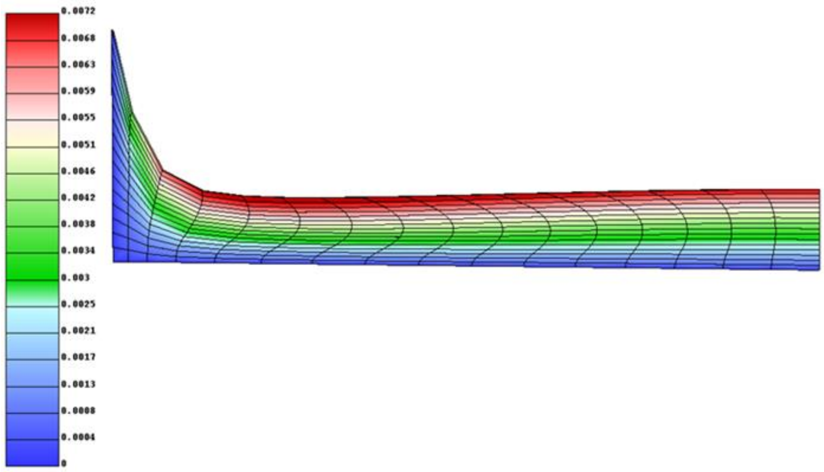

Figure 2. Vertical displacements field of the strip.

5. Conclusion

Numerical algorithm of investigation of hyperelastic solids using Finger strain measure is described. Physical equations are defined using strain potential function. For Mooney–Rivlin potential linearized constitutive relations and resolving expression are obtained. Numerical implementation is based on finite element method using 8-node brick element.

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