On solvability of a elliptic-parabolic problem of nonlinear filtration theory

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Abstract. Existence of a strong solution of the initial-boundary value problem modeling the process of liquid filtration in an arbitrary bounded region Ω of space R^n is proven. For determining a generalized solution, the Kirchhoff transform is used, and it is assumed that the domain of the Kirchhoff function constitutes only a part of the real axis. For proving the existence theorem, the method of semidiscretization with respect to the variable *t* and the Galerkin method are used.

1. Introduction

In the present work, a proof is conducted for existence of a strong solution of the initial-boundary value problem governed by the equation of nonlinear non-stationary filtration of the following type:

$$m\frac{\partial s(p)}{\partial t} + \operatorname{div} q = f\left(s(p)\right). \tag{1}$$

Here p is pore pressure, q is filtration rate, m is porosity of the medium, $s(p)$ is saturation. We will assume that dependence of filtration rate q on p is given by the following law:

$$q = -b\left(s(p)\right)\left(\nabla p - e\right), \tag{2}$$

where the function $b\left(s(p)\right)$ is relative phase permeability of the medium, $e = \rho g$, ρ is liquid density, g is gravitational force vector.

The equation under consideration is of the variable type: it is elliptic in the saturated filtration region (at $p \geq 0$), as in this case $s(p) = 1$, and parabolic at $p < 0$.

The problem of saturated-unsaturated filtration has been the subject of research by many authors. Most of the published works are devoted to the construction of approximate methods for solve this problem and discuss the results of numerical experiments (see, for example: [1]–[5]). Publications theoretical are few, the most important are [6]–[8], which discuss the existence of a generalized solution, and [10]–[11] devoted to the study of uniqueness of the solution.

Describe the results of [6]–[8].

First, we note that [6] is the fundamental work. Developed by the authors of this article a technique is widely used in the study of the solvability of nonlinear time-dependent problems of mathematical physics, in particular, it was used in the current work.

In [6] for a system of nonlinear equations of elliptic-parabolic type with strongly monotone space operator theorems on the existence of a generalized solution have been proved and studied its properties. This type of equation arises if in (1) by using Kirchhoff transformation we pass from an unknown function $p(x,t)$ to the function $u(x,t)=\vartheta(p(x,t))$, where

$$\vartheta(p)=\int_0^p b(s(\xi))\,d\xi. \tag{3}$$

Solvability of the equation resulted from this transformation follows from [6], provided that the domain of the Kirchhoff function ϑ was the entire axis of real numbers, and $p \leftrightarrow \vartheta$ was a one-to-one mapping.

It was later noted that the assumption that the domain of function ϑ represents the entire axis of real numbers is quite limited. We demonstrate this using conventional for applied research dependences $s(p)$ and $b(s)$:

$$s(p)=\begin{cases}(1-p)^{-\alpha}, & p\le 0\\ 1, & p>0,\end{cases} \qquad b(s)=s^{\beta},\quad 0\le s\le 1,\quad \alpha>0,\quad \beta>0,\quad \alpha\beta>1. \tag{4}$$

Apparently, in this case we have $\vartheta(p)\in[-\gamma,+\infty) \quad \forall p\in R$, where $\gamma=\int_{-\infty}^{0} b(s(\xi))\,d\xi=(\alpha\beta-1)^{-1}$.

In problems of filtration theory, the number $\int_{-\infty}^{0} b(s(\xi))\,d\xi$ is referred to as the macroscopic capillary length, which, as it was proven in [9], is always bounded.

The authors of [7], [8] consider the case when the domain of function ϑ is semi-axis of real numbers $[-\gamma,+\infty)$. In [7] is proved the existence theorem for weak solution in the one-dimensional case. In [8] the problem of saturated-unsaturated filtration in case when $b(s)=0$, if $s\le\hat{s}$, where $\hat{s}$ is minimum irreducible wetness of soil was considered. In that work, a proof was given for existence of a solution of a regularized problem obtained from the original problem by substitution of the function $\tilde{\varphi}(u)=s\left(\vartheta^{(-1)}(u)\right)$ on set $[-\gamma,-\gamma+\delta(\varepsilon)]$ (it is a zone of unbounded increase of the function $\tilde{\varphi}$) by a linear function having an angle with the axis u, the tangent of which is equal to $1/\varepsilon,$ where ε is a small parameter.

The aim of the present work is to prove existence of a generalized solution of problem (1)–(2) under more general assumptions on the functions $s(p)$ and $b(s)$, which, in particular, also permit dependencies of the form (4).

2. Problem statement

We consider problem (1), (2) in bounded region Ω of space R^n with Lipschitz boundary $\Gamma=\Gamma_1\cup\Gamma_2$ assuming that, at $t\in(0,T]$, the boundary conditions

$$q_\nu(x,t)=0,\quad x\in\Gamma_1; \qquad p(x,t)=0,\quad x\in\Gamma_2 \tag{5}$$

are satisfied.

Here $q_\nu(x,t)=q\cdot\nu$, where ν is outward normal to boundary Γ, $q\cdot\nu$ is scalar product of vectors q and ν in space R^n. Initial conditions are set in the form

$$\left(p(x,0)\right)^{-}=p_0^{-}(x),\qquad x\in\Omega, \tag{6}$$

where $p^- = (|p| - p)/2$, p_0 is given function.

We use the Kirchhoff transform for proving the existence theorem. To eliminate the constraint $u \geq -\gamma$, for given $s(p), b(s)$, $\vartheta(p)$ we introduce the function (see (3))

$$\varphi(u) = \begin{cases} s\left(\vartheta^{-1}(u)\right), & u > -\gamma, \\ 0, & u \leq -\gamma. \end{cases} \tag{7}$$

Along with (1), (2), (5), (6) we consider the problem (which is assigned the name “extended” below) that is obtained from (1), (2), (5), (6) by replacement of equations (1), (2), boundary conditions (5) and initial conditions (6) with the relations:

$$m\frac{\partial \varphi(u)}{\partial t} + \operatorname{div} q = f\left(\varphi(u)\right), \tag{8}$$

$$q = -\left(\nabla u - b\left(\varphi(u)\right)e\right), \tag{9}$$

$$q_\nu(x,t) = 0, \quad x \in \Gamma_1; \qquad u(x,t) = 0, \quad x \in \Gamma_2, \tag{10}$$

$$u_0^- = \left(u(x,0)\right)^- = \vartheta\left(p_0^-(x)\right). \tag{11}$$

One can readily see that if function u is a solution of the extended problem, and $u(x,t) \geq -\gamma$ for arbitrary $(x,t) \in Q_T$, then $\vartheta^{-1}(u)$ is a solution of the problem (1), (2), (5), (6).

From this point on, we assume that the functions $s(p), b(s)$ satisfy the following conditions:

A$_1$. Function $s: R \to (0,1]$ is nondecreasing, continuous and piecewise-differentiable and satisfies the relation $s(p) = 1 \;\; \forall p \geq 0$.

A$_2$. Function $b: [0,1] \to [0,1]$ is nondecreasing and continuous, $b(0) = 0$.

A$_3$. Mapping $\vartheta: R \to [-\gamma, +\infty)$, given by formula (3), is one-to-one, where $\gamma = \int\limits_{-\infty}^{0} b\left(s(\xi)\right) d\xi$.

A$_4$. There exist constants $\kappa > 0$, $C_\varphi > 0$ such that

$$\varphi'(\eta)\varphi^{2\kappa-1}(\eta) \leq C_\varphi \qquad \forall \eta \in (-\gamma, 0). \tag{12}$$

Let us note that for dependencies (4), the condition (12) holds at $\kappa > \beta/2$ for arbitrary α and β satisfying the condition $(\alpha\beta - 1) > 0$.

3. Existence theorem

Let V be a subspace of the space $W_2^1(\Omega)$ having a zero trace on Γ_2.

Definition 1. *A generalized solution of the extended problem is a function* $u \in L_2(0,T;V)$, *for which the conditions*

$$\frac{\partial \varphi(u)}{\partial t} \in L_2\left(0,T;V^*\right), \qquad u^-(x,0) = u_0^-(x) \quad \text{almost everywhere on } \Omega$$

are satisfied, and for an arbitrary function $z \in L_2(0,T;V)$, *the equality*

$$\int\limits_0^T < m\frac{\partial \varphi}{\partial t}, z > dt + \int\limits_{Q_T} \nabla u \cdot \nabla z \, dxdt = \int\limits_{Q_T} \left\{ b\left(\varphi(u)\right)\left(e \cdot \nabla z\right) + f\left(\varphi(u)\right)z \right\} dxdt \tag{13}$$

holds. Here $Q_T = (0,T) \times \Omega$, V^* *is space dual to* V, $\langle v, z \rangle$ *is a value of the functional* $v \in V^*$ *on the element* $z \in V$.

Theorem 1. *Let the functions $s(p), b(s)$ and $\varphi(u)$ satisfy the conditions **A_1–A_4**. The function $f(\xi)$ is continuous on $[0,1]$ and $f(0)=0$. Then, for arbitrary $u_0 \in V$, there exists a generalized solution of the extended problem.*

For proving the theorem, we use the semidiscretization method. On the interval $[0,T]$, a uniform grid is created having the step size

$$\overline{\omega}_\tau = \{t = k\tau,\, 0 \le k \le M,\, M\tau = T\}, \qquad \omega_\tau = \overline{\omega}_\tau \setminus \{0\}.$$

Definition 2. *The function $w_\tau(t)$ is a semidiscrete solution of the extended problem, if*

$$u_\tau(t) \in V \qquad \forall t \in \omega_\tau, \qquad (u_\tau(0))^- = u_0^- \ \text{ almost everywhere on } \Omega,$$

for any function $z \in V$ and for any $t \in \overline{\omega}_\tau \setminus \{T\}$, the equalities

$$\int_\Omega m \frac{\varphi(\hat{u}_\tau) - \varphi(u_\tau)}{\tau} z dx + \int_\Omega \nabla \hat{u}_\tau \cdot \nabla z dx = \int_{Q_T} \{b(\varphi(\hat{u}_\tau))(e \cdot \nabla z) + f(\varphi(\hat{u}_\tau)) z\} dxdt \tag{14}$$

hold. Here $\hat{y} = y(t+\tau)$.

Lemma 1. *For a solution of the semidiscrete problem (14), the inequality*

$$u_\tau(x,t) \ge -\gamma \qquad \text{almost everywhere on } \Omega, \qquad \forall t \in \omega_\tau \tag{15}$$

holds.

Proof. By setting $z = (\gamma + \widehat{w}_\tau)^-$ in equality (14), we rewrite it in the form

$$\frac{m}{\tau}\int_\Omega \varphi(\hat{u}_\tau)(\gamma + \hat{u}_\tau)^- \, dx - \frac{m}{\tau}\int_\Omega \varphi(u_\tau)(\gamma + \hat{u}_\tau)^- \, dx + \int_\Omega \nabla \hat{u}_\tau \cdot \nabla(\gamma + \hat{u}_\tau)^- dx$$
$$= \int_{Q_T} \{b(\varphi(\hat{u}_\tau))(e \cdot \nabla(\gamma + \hat{u}_\tau)^-) + f(\varphi(\hat{u}_\tau))(\gamma + \hat{u}_\tau)^-\} dxdt. \tag{16}$$

One can readily see that, by virtue of equality (7), the function $\varphi(\zeta)$ is zero for $\zeta \le -\gamma$, and the relation $(\gamma + \zeta)^- = 0$ holds for $\zeta \ge -\gamma$; therefore, $\varphi(\zeta)(\gamma+\zeta)^- = 0$ and $\zeta^+(\gamma + \zeta)^- = 0$ for arbitrary values of ζ. In addition, from conditions $b(0) = 0$ and $f(0) = 0$ it follows that $b(\varphi(\zeta)) \ne 0$ and $f(\varphi(\zeta)) \ne 0$ if and only if $\zeta > -\gamma$. Therefore, the right-hand side (16) is equal to zero.

Let us transform the third term on the left-hand side in relation (16):

$$\int_\Omega \nabla \hat{u}_\tau \cdot \nabla(\gamma + \hat{u}_\tau)^- \, dx = -\int_\Omega \sum_{i=1}^n \frac{\partial}{\partial x_i}(\gamma + \hat{u}_\tau) \frac{\partial}{\partial x_i}(\gamma + \hat{u}_\tau)^- \, dx = -\|(\gamma + \hat{u}_\tau)^-\|_1^2. \tag{17}$$

Then equality (16) takes the form

$$-\|(\gamma + \hat{u}_\tau)^-\|_1^2 - \frac{m}{\tau}\int_\Omega \varphi(u_\tau)(\gamma + \hat{u}_\tau)^- \, dx = 0, \tag{18}$$

here $\|v\|_1^2 = \int_\Omega |\nabla v|^2 \, dx.$ The left-hand side in equality (18) is nonpositive; therefore, it follows from (18) that $(\gamma + \hat{u}_\tau(x,t))^- = 0$ almost everywhere in Ω $\forall t \in \omega_\tau$. The proof of the lemma is completed.

We use the Galerkin method to prove existence of a semidiscrete solution, for which a priori estimates of the kind

$$\int_\Omega \Phi(u_\tau(t'))\, dx \le C, \qquad \sum_{t=0}^{t'-\tau} \tau \|\hat{u}_\tau(t)\|_1^2 \le C, \qquad \sum_{t=0}^{t'-\tau} \tau \|G(\hat{u}_\tau(t))\|_1^2 \le C \qquad \forall t' \in \omega_\tau, \tag{19}$$

$$\frac{1}{k\tau}\sum_{t'=0}^{T-k\tau}\tau\int_{\Omega}\left(G(u_\tau(t'+k\tau))-G(u_\tau(t'))\right)^2 dx \le C \tag{20}$$

are satisfied, where $\Phi(u)=\int_0^u \varphi'(\xi)\xi\, d\xi$, $G(u_\tau)=\int_0^{u_\tau}\varphi'(\xi)g(\xi)\, d\xi$,

$$g(\zeta)=\begin{cases}\varphi^{2\kappa}(\zeta)\left(1+\varphi^{2\kappa}(\zeta)\right)^{-1}, & \text{if } \zeta<0,\\ 1/2, & \text{if } \zeta\ge 0.\end{cases}$$

Introducing the function $G(u)$ and obtaining the estimates represent an important stage of a theorem's proof as the estimates (19), (20) ensure existence of a converging subsequence of functions u_τ^- almost everywhere in Q_T (see [6]). For investigating non-linear problems, existence of such convergence is essential. It is allocated in the proof of the Theorem 1 a sequence of piecewise constant in t extension of solutions of a semidiscrete problem. It is proved that limit of this sequence is a generalized solution of the extended problem.

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