Modified Coulomb and Lorenz gauges in the modeling of low-frequency electromagnetic processes

A V Kalinin, A A Tiukhtina and S R Lavrova

Lobachevsky State University of Nizhnii Novgorod, 23 Gagarin av., 603950 Nizhnii Novgorod, Russia

E-mail: kalinmm@yandex.ru

Abstract. The boundary value problem for the quasistationary magnetic approximation of the time-harmonic Maxwell equations in inhomogeneous media is studied. The considered problem is reduced to the variational problem of determining vector magnetic and scalar electric potentials. The special gauges are discussed, that generalize the Coulomb and Lorenz gauges and allow to formulate the problems of the independent definitions of the vector magnetic potential. The correctness of the problems are established under general conditions on the coefficients. The relation between solutions of the problems with different gauges is studied. The equivalence of the problems for potentials to the original boundary value problem is proved.

1. Introduction

The solution of many actual technological problems leads to theoretical and numerical study of problems for quasi-stationary electromagnetic fields in physically heterogeneous media [1]. Quasi-stationary magnetic or low-frequency approximation of the Maxwell's equations used in the design of electromagnetic devices in metallurgical industry and power transformers, for remote sensing, in solving problems of magnetic levitation and in advanced medicine [1]–[3].

The essential stage in the development and justification of efficient numerical methods for solving applied problems is the study questions of the correctness of their various mathematical formulations. Electromagnetic processes in theoretical and numerical researches are traditionally described in terms of magnetic or electric fields [4]-[6] either in terms of potentials – vector electrical and scalar magnetic (the $\boldsymbol{T}-\psi$ formulations) [7]-[9] or vector magnetic and scalar electrical (the $\boldsymbol{A}-\varphi$ formulations) [10]-[16].

The application of potentials always presupposes the choosing of certain gauges that ensure the uniqueness of the solution of the problem. In particular, for the description of stationary and quasistationary electromagnetic fields the classical Coulomb gauge $\operatorname{div}\boldsymbol{A}=0$ and Lorenz gauge $\operatorname{div}\boldsymbol{A}+\mu\sigma\varphi=0$ are used [11]-[14], which allow to uniquely determine the vector magnetic and scalar electric potentials.

In the case of inhomogeneous media the classic Coulomb and Lorenz gauges do not lead to the decoupled problem for the electric and magnetic potentials [13], [14], however this decoupling plays an important role in the algorithms development for the numerical solution. In this regard, various modifications of the gauges are considered. In [14], [15] the possibility of applying the modified

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Lorenz gauge, that allows to separate the problems of determining the magnetic and electric potentials, is discussed.

In the present paper the theoretical issues are investigated related to the use of the modified gauges, which generalize the Coulomb and Lorenz gauges for the case of inhomogeneous media and decouple the problem for the vector magnetic and scalar electric potentials. We consider the time - harmonic Maxwell equations in the low frequency approximation in physical domains filled with conductive and non-conductive material. The respective topics for conductors are considered for the stationary problems in [17], [18], for time-dependent problems – in [19].

For the considered class of problems this paper provides a full justification of the suitability of these gauges, namely, we prove the well-posedness of the problems and their equivalence to the original boundary value problem. The relation between solutions of the problems under different gauges is examined. In the case of homogeneous media, this relation is similar to the ratio between the solutions for the system of Stokes equations for incompressible and weakly compressible liquid [20].

The study of the properties of the boundary value problems is based on the inequalities which combine the scalar product of vector fields and norms of their curl and divergence. These inequalities generalize the estimates for scalar products of vector fields obtained in [17], [21] for the case of domains homeomorphic to a ball.

2. The boundary value problem

The time-harmonic Maxwell equations in the quasistationary magnetic approximation can be presented in the form [1]

$$\operatorname{curl}\boldsymbol{H}(x)=\boldsymbol{J}(x), \tag{1}$$

$$\operatorname{div}\boldsymbol{B}(x)=0, \tag{2}$$

$$\operatorname{curl}\boldsymbol{E}(x)=-\mathrm{i}\,\omega\boldsymbol{B}(x), \tag{3}$$

$$\operatorname{div}\boldsymbol{D}(x)=\rho(x), \tag{4}$$

where $x\in\Omega\subset\mathbf{R}^3$; $\boldsymbol{H}$, $\boldsymbol{B}$, $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{J}:\Omega\to\mathbf{C}^3$ and $\rho:\Omega\to\mathbf{C}^1$ are unknown functions.

In linear media the following constitutive relations are valid:

$$\boldsymbol{B}=\mu\boldsymbol{H},\ \boldsymbol{D}=\varepsilon\boldsymbol{E},\ \boldsymbol{J}=\sigma\boldsymbol{E}+\boldsymbol{J}^{ext}, \tag{5}$$

where μ is a magnetic permeability, ε is a permittivity, σ is a electrical conductivity, $\boldsymbol{J}^{ext}$ is an exterior current density.

We suppose $\omega\ge 0$. In case $\omega=0$ let $\boldsymbol{H}$, $\boldsymbol{B}$, $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{J}$, ρ are real valued functions and thus (1)-(4) is the stationary system of the Maxwell equations.

It is assumed in this paper, that Ω is an open bounded domain, homeomorphic to a ball, with a Lipschitz boundary $\partial\Omega$. Let $\boldsymbol{\nu}(x)$ is the unit normal vector in $x\in\partial\Omega$. For function $\boldsymbol{u}:\overline{\Omega}\to\mathbf{C}^3$ we denote by $\boldsymbol{u}_\nu$, $\boldsymbol{u}_\tau$ the normal and tangent components of $\boldsymbol{u}$ on $\partial\Omega$.

The system (1)–(5) is considered with the boundary condition

$$\boldsymbol{H}_\tau(x)=0,\ x\in\partial\Omega. \tag{6}$$

We assume, the domain consists from the conductor Ω_C and the isolator $\Omega_I=\Omega\setminus\overline{\Omega}_C$. Let Ω_C is an open bounded homeomorphic to a boll domain with Lipschitz boundary Γ, $\overline{\Omega}_C\subset\Omega$, so $\partial\Omega_I=\partial\Omega\cup\Gamma$. The unit normal vectors in $x\in\Gamma$ to Ω_C and Ω_I are denoted by $\boldsymbol{\nu}_C(x)$ and $\boldsymbol{\nu}_I(x)$ respectively, $\boldsymbol{\nu}_C(x)+\boldsymbol{\nu}_I(x)=0$. For functions $\boldsymbol{u}:\Omega\to\mathbf{C}^3$ and $u:\Omega\to\mathbf{C}^1$ we denote by $\boldsymbol{u}_C$, u_C their restrictions on Ω_C, by $\boldsymbol{u}_I$, u_I their restrictions on Ω_I.

The ε and μ are self-ajoint linear operators from $\{L_2(\Omega)\}^3$ into $\{L_2(\Omega)\}^3$, satisfying the following conditions:

$$\varepsilon_1\|\boldsymbol{u}\|_{2,\Omega}^2 \le (\varepsilon\boldsymbol{u},\boldsymbol{u})_{2,\Omega} \le \varepsilon_2\|\boldsymbol{u}\|_{2,\Omega}^2,\ \mu_1\|\boldsymbol{u}\|_{2,\Omega}^2 \le (\mu\boldsymbol{u},\boldsymbol{u})_{2,\Omega} \le \mu_2\|\boldsymbol{u}\|_{2,\Omega}^2,$$

$\sigma = \sigma(x)$ is symmetric 3×3 matrix of measurable functions on Ω, satisfying the conditions

$$\sigma_{ij}(x) = 0,\ i,j = 1,2,3, \text{ for almost all } x \in \Omega_I,$$

$$\sigma_1|\xi|^2 \le (\sigma(x)\xi,\xi) \le \sigma_2|\xi|^2 \text{ for almost all } x \in \Omega_C \text{ and for all } \xi \in \mathbf{R}^3,$$

where ε_i, μ_i, σ_i, ($i = 1,2$) are given positive numbers, by $\|\cdot\|_{2,\Omega}$ and $(\cdot,\cdot)_{2,\Omega}$ the norm and the scalar product in $\{L_2(\Omega)\}^3$ are denoted.

$\boldsymbol{J}^{ext}: \Omega \to \mathbf{C}^3$ is a square integrable function, such that

$$\operatorname{div}\boldsymbol{J}_I^{ext} = 0,\ \left(\boldsymbol{J}_I^{ext}\right)_\nu(x) = 0,\ x \in \partial\Omega.$$

The generalized solutions of the problems will be considered, that is all equalities have to be satisfied in the sense of distributions and boundary conditions have to be satisfied in the sense of the trace theory [22].

The following Hilbert spaces with the respective scalar products are defined [22]:

$$H(\mathrm{div};\Omega) = \left\{\boldsymbol{u} \in \{L_2(\Omega)\}^3 : \operatorname{div}\boldsymbol{u} \in L_2(\Omega)\right\},\ K(\mathrm{div};\Omega) = \left\{\boldsymbol{u} \in \{L_2(\Omega)\}^3 : \operatorname{div}\boldsymbol{u} = 0\right\},$$

$$(\boldsymbol{u},\boldsymbol{v})_{div,\Omega} = \int_\Omega (\boldsymbol{u}\cdot\bar{\boldsymbol{v}})\mathrm{d}x + \int_\Omega \operatorname{div}\boldsymbol{u}\operatorname{div}\bar{\boldsymbol{v}}\mathrm{d}x,$$

$$H(\mathrm{curl};\Omega) = \left\{\boldsymbol{u} \in \{L_2(\Omega)\}^3 : \operatorname{curl}\boldsymbol{u} \in \{L_2(\Omega)\}^3\right\},\ K(\mathrm{curl};\Omega) = \left\{\boldsymbol{u} \in \{L_2(\Omega)\}^3 : \operatorname{curl}\boldsymbol{u} = 0\right\},$$

$$(\boldsymbol{u},\boldsymbol{v})_{curl,\Omega} = \int_\Omega (\boldsymbol{u}\cdot\bar{\boldsymbol{v}})\mathrm{d}x + \int_\Omega (\operatorname{curl}\boldsymbol{u}\operatorname{curl}\bar{\boldsymbol{v}})\mathrm{d}x.$$

$H_0(\mathrm{div};\Omega)$, $H_0(\mathrm{curl};\Omega)$ denote the closures of the set of test vector-functions in $H(\mathrm{div};\Omega)$ and $H(\mathrm{curl};\Omega)$ respectively, $K_0(\mathrm{div};\Omega) = K(\mathrm{div};\Omega) \cap H_0(\mathrm{div};\Omega)$.

The solution of the problem (1)–(6) is the set of functions $\boldsymbol{H} \in H_0(\mathrm{curl};\Omega)$, $\boldsymbol{B} \in K(\mathrm{div};\Omega)$, $\boldsymbol{J} \in K_0(\mathrm{div};\Omega)$, $\boldsymbol{E} \in H(\mathrm{curl};\Omega)$, $\boldsymbol{D} \in \{L_2(\Omega)\}^3$, $\rho \in H^{-1}(\Omega)$, satisfying (1), (3)–(5).

To uniquely identify function $\boldsymbol{E}$ in the whole domain Ω, we suppose that unknown functions $\boldsymbol{E}$, ρ satisfy the conditions

$$\rho_I = \rho_0,\ (\varepsilon\boldsymbol{E})_\nu(x) = 0,\ x \in \partial\Omega, \tag{7}$$

where $\rho_0 \in L_2(\Omega_I)$ is a given function.

In case $\omega = 0$ the problem also may be regarded under conditions

$$\rho_I = \rho_0,\ \boldsymbol{E}_\tau(x) = 0,\ x \in \partial\Omega,\ \left\langle(\varepsilon\boldsymbol{E})_{\nu_I},1\right\rangle_\Gamma = Q, \tag{8}$$

Q is a given constant.

3. Problems for the vector potential

Relations (2), (3) allow to introduce the vector magnetic potential $\boldsymbol{A}$ and the scalar electric potential φ as new unknown variables by the formulas [23]

$$\boldsymbol{B} = \operatorname{curl} \boldsymbol{A}, \ \boldsymbol{E} = -\operatorname{grad} \varphi - \mathrm{i}\,\omega \boldsymbol{A} \tag{9}$$

In this case, the system (1)–(5) is reduced to one equation

$$\mathrm{i}\,\omega\sigma\boldsymbol{A} + \operatorname{curl} \mu^{-1} \operatorname{curl} \boldsymbol{A} = -\sigma \operatorname{grad} \varphi + \boldsymbol{J}^{ext}. \tag{10}$$

Equation (10) is provided by the boundary condition corresponding to (6):

$$\left(\mu^{-1} \operatorname{curl} \boldsymbol{A}\right)_\tau (x) = 0, \ x \in \partial\Omega. \tag{11}$$

The solution of the problem (10), (11) is the functions $\boldsymbol{A} \in H(\mathrm{curl};\Omega)$, $\varphi \in H^1(\Omega)$, satisfying (10) in the sense of the distribution and (11) in the sense of the trace theory, that is $\mu^{-1} \operatorname{curl} \boldsymbol{A} \in H_0(\mathrm{curl};\Omega)$.

Suppose that $\boldsymbol{A}$, φ is the solution of (10), (11). Then from (10) we obtain for all $\boldsymbol{v} \in H(\mathrm{curl};\Omega)$

$$\mathrm{i}\omega \int_{\Omega_C} (\sigma\boldsymbol{A} \cdot \bar{\boldsymbol{v}})\mathrm{d}x + \int_{\Omega} \left(\mu^{-1} \operatorname{curl} \boldsymbol{A} \cdot \operatorname{curl} \bar{\boldsymbol{v}}\right)\mathrm{d}x = -\int_{\Omega_C} (\sigma \operatorname{grad} \varphi \cdot \bar{\boldsymbol{v}})\mathrm{d}x + \int_{\Omega} \left(\boldsymbol{J}^{ext} \cdot \bar{\boldsymbol{v}}\right)\mathrm{d}x. \tag{12}$$

The solution of the problem (10), (11) is obviously not unique. The following two types of gauges are discussed:

$$\operatorname{div} \sigma\boldsymbol{A}_C = 0, \ (\sigma\boldsymbol{A})_{\nu_C}(x) = 0, \ x \in \Gamma, \ \operatorname{div} \boldsymbol{A}_I = 0, \ \boldsymbol{A}_\nu(x) = 0, \ x \in \partial\Omega, \tag{13}$$

and

$$\varphi_C = -\kappa \operatorname{div} \sigma\boldsymbol{A}_C, \ (\sigma\boldsymbol{A})_{\nu_C}(x) = 0, \ x \in \Gamma, \ \operatorname{div} \boldsymbol{A}_I = 0, \ \boldsymbol{A}_\nu(x) = 0, \ x \in \partial\Omega, \tag{14}$$

where $\kappa > 0$ is an arbitrary constant.

We define the Gilbert spaces

$$W(\sigma;\Omega_C,\Omega) = \{\boldsymbol{u} \in H(\mathrm{curl};\Omega) : \sigma\boldsymbol{u}_C \in H_0(\mathrm{div};\Omega_C), \operatorname{div} \boldsymbol{u}_I = 0, (\boldsymbol{u}_I)_\nu(x) = 0, x \in \partial\Omega\},$$

$$(\boldsymbol{u},\boldsymbol{v})_W = \int_{\Omega} (\boldsymbol{u} \cdot \bar{\boldsymbol{v}})\mathrm{d}x + \int_{\Omega} (\operatorname{curl} \boldsymbol{u} \cdot \operatorname{curl} \bar{\boldsymbol{v}})\mathrm{d}x + \int_{\Omega_C} (\operatorname{div} \sigma\boldsymbol{u} \cdot \operatorname{div} \sigma\bar{\boldsymbol{v}})\mathrm{d}x,$$

$$V(\sigma;\Omega_C,\Omega) = \{\boldsymbol{u} \in W(\sigma;\Omega_C,\Omega) : \operatorname{div} \sigma\boldsymbol{u}_C = 0\}, \ (\boldsymbol{u},\boldsymbol{v})_V = (\boldsymbol{u},\boldsymbol{v})_{curl,\Omega}.$$

Using (12) we obtain that the problem (10), (11), (13) is reduced to the following problem: to find a function $\boldsymbol{A} \in V(\sigma;\Omega_C,\Omega)$ such that for all $\boldsymbol{v} \in V(\sigma;\Omega_C,\Omega)$

$$\mathrm{i}\omega \int_{\Omega_C} (\sigma\boldsymbol{A} \cdot \bar{\boldsymbol{v}})\mathrm{d}x + \int_{\Omega} \left(\mu^{-1} \operatorname{curl} \boldsymbol{A} \cdot \operatorname{curl} \bar{\boldsymbol{v}}\right)\mathrm{d}x = \int_{\Omega} \left(\boldsymbol{J}^{ext} \cdot \bar{\boldsymbol{v}}\right)\mathrm{d}x, \tag{15}$$

the problem (10), (11), (14) is reduced to the problem of determining $\boldsymbol{A} \in W(\sigma;\Omega_C,\Omega)$ such that for all $\boldsymbol{v} \in W(\sigma;\Omega_C,\Omega)$

$$\mathrm{i}\omega \int_{\Omega_C} (\sigma\boldsymbol{A} \cdot \bar{\boldsymbol{v}})\mathrm{d}x + \int_{\Omega} \left(\mu^{-1} \operatorname{curl} \boldsymbol{A} \cdot \operatorname{curl} \bar{\boldsymbol{v}}\right)\mathrm{d}x + \kappa \int_{\Omega_C} (\operatorname{div} \sigma\boldsymbol{A} \cdot \operatorname{div} \sigma\bar{\boldsymbol{v}})\mathrm{d}x = \int_{\Omega} \left(\boldsymbol{J}^{ext} \cdot \bar{\boldsymbol{v}}\right)\mathrm{d}x. \tag{16}$$

Theorem 1. *The problems (15) and (16) have unique solutions*.

The next theorem establishes the relation between the solutions of the problems under different gauges.

Theorem 2. *Let* $\boldsymbol{A}$ *and* $\boldsymbol{A}_\kappa$ *are the solutions of (15) and (16) respectively. Then* $\operatorname{curl} \boldsymbol{A} = \operatorname{curl} \boldsymbol{A}_\kappa$, $\boldsymbol{A}_\kappa \to \boldsymbol{A}$ *in* $W(\sigma;\Omega_C,\Omega)$ *as* $\kappa \to \infty$ *and the following estimation is valid*

$$\left\| \boldsymbol{A}_\kappa - \boldsymbol{A} \right\|_W \le C\kappa^{-1} \left\| \boldsymbol{J}^{ext} \right\|_{2,\Omega},$$

where the constant $C > 0$ *depends only on* Ω_C *and* Ω.

The proof of the theorems is based on the following inequalities which generalize the obtained in [18], [22] estimates for scalar products of vector fields in a star-shape domain.

Theorem 3. *Let* $\Omega \subset \mathbf{R}^3$ *is an open bounded Lipschitz domain, homeomorphic to a ball. There exists a positive constant* $C(\Omega)$, *which depends only on* Ω, *such that the inequality*

$$\left|(\boldsymbol{u},\boldsymbol{v})_{2,\Omega}\right| \le C(\Omega)\left(\|\boldsymbol{u}\|_{2,\Omega}\|\operatorname{div}\boldsymbol{v}\|_{2,\Omega} + \|\operatorname{curl}\boldsymbol{u}\|_{2,\Omega}\|\boldsymbol{v}\|_{2,\Omega} + \|\operatorname{curl}\boldsymbol{u}\|_{2,\Omega}\|\operatorname{div}\boldsymbol{v}\|_{2,\Omega}\right)$$

holds for any $\boldsymbol{u} \in H_0(\mathrm{curl};\Omega)$, $\boldsymbol{v} \in H(\mathrm{div};\Omega)$ *and for any* $\boldsymbol{u} \in H(\mathrm{curl};\Omega)$, $\boldsymbol{v} \in H_0(\mathrm{div};\Omega)$.

4. The correctness of application the gauges

The using of potentials requires that $\boldsymbol{H}$, $\boldsymbol{E}$, which defined by (9), satisfy the Maxwell equations. Thus the correctness of application the gauges implies the equivalence the problems in terms of potentials and the origin boundary value problem.

The conditions (7) in terms of potentials means

$$\operatorname{div}\varepsilon(\mathrm{i}\omega\boldsymbol{A} + \operatorname{grad}\varphi)_I = -\rho_0, \ \varepsilon(\mathrm{i}\omega\boldsymbol{A} + \operatorname{grad}\varphi)_\nu(x) = 0, \ x \in \partial\Omega. \tag{17}$$

The conditions (8) for $\omega = 0$ means

$$\operatorname{div}\varepsilon\operatorname{grad}\varphi_I = -\rho_0, \ \varphi(x) = \mathrm{const}, \ x \in \partial\Omega, \ \left\langle(\varepsilon\operatorname{grad}\varphi)_{\nu_I}, 1\right\rangle_\Gamma = -Q. \tag{18}$$

The following statements are valid.

Theorem 4. *Let* $\boldsymbol{A} \in W(\sigma;\Omega_C,\Omega)$ *is the solution of the problem (15) or (16). Then there is a function* $\varphi \in H^1(\Omega)$, *unique up to an additive constant, such that* $\boldsymbol{A}$, φ *is the solution of the problem (10), (11), (17). In case* $\omega = 0$ *there is a unique function* $\varphi \in H_0^1(\Omega)$ *such that* $\boldsymbol{A}$, φ *is the solution of the problem (10), (11), (18).*

Theorem 5. *Let* $\boldsymbol{A} \in W(\sigma;\Omega_C,\Omega)$ *is the solution of the problem (16). Then there is a unique function* $\varphi \in H^1(\Omega)$, *such that* $\boldsymbol{A}$, φ *is the solution of the problem (10), (11), (17) and* $\varphi_C = -\kappa\operatorname{div}\sigma\boldsymbol{A}_C$. *In case* $\omega = 0$ *there is a unique function* $\varphi \in H^1(\Omega)$, *such that* $\boldsymbol{A}$, φ *is the solution of the problem (10), (11), (18) and* $\varphi_C = -\kappa\operatorname{div}\sigma\boldsymbol{A}_C$.

From theorems 4, 5 the next theorem follows

Theorem 6. *The problems (1)-(7) and (1)-(6), (8) (at* $\omega = 0$*) have unique solutions. Moreover the relations (9) are valid, where* $\boldsymbol{A}$ *is the solution of (15) or (16).*

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References

[1] Alonso Rodriguez A and Valli A 2010 *Eddy current approximation of Maxwell equations* (Milan: Springer-Verlag Italia)

[2] Galanin M P and Popov Y P 1995 *Kvazistationarnye elektromagnitnye polya v neodnorodnykh*

IOP Conf. Series: Materials Science and Engineering **158** (2016) 012046 doi:10.1088/1757-899X/158/1/012046

sredah (Moscow: Nauka)
[3] Kalinin A V, Sumin M I and Tyukhtina A A 2016 *Differential Equations* **52 (5)** 1–17
[4] Meddahi S and Selgas V 2008 *App. Num. Math.* **58** 1061–1083
[5] Prato Torres R A, Stephan E P and Leydecker F 2010 *Computing* **88** 131–154
[6] Arnold L and Harrach B 2012 *SIAM J. Appl. Math.* **72 (2)** 558–576
[7] Albanese R and Rubinacci G 1998 Finite Element Methods for the Solution of 3D Eddy Current Problems *Advances in Imaging and Electron Physics* **102** ed. P W Hawkes (San Diego: Academic Press) pp 2–86
[8] Chen T, Kang T, Lu G and Wu L 2014 *Math. Meth. Appl. Sci.* **37** 343–359
[9] Preis K, Biro O and Ticar I 2000 *IEEE Trans. Magn.* **36** 840–843
[10] Fernandes P 1995 *IEE Proc.-Sci. Meas. Technol.* **142** 299–306
[11] Coulomb J-L and Sabonnadiere J-C 1985 *CAO en electrotechnique* (Paris: Hermes Publ.)
[12] Jackson J D 2002 *Am.J.Phys* **70** (9) 917–928.
[13] Biro O and Valli A 2007 *Comput. Meth. Appl. Mech. Engrg*. 2007 **196** 890–1904
[14] Fernandes P and Valli A 2008 *Math. Meth. Appl. Sci.* **31** 71–98
[15] Bossavit A 1999 *COMPEL* **18 (**3) 323–336
[16] Kang T and Kim K I 2009 *Computing* **85** 339–362
[17] Kalinin A V and Kalinkina A A 2002 *Vestnik Nizhegorod. Gos. Univ. Mat. Model. Optim. Upravl.* **25 (1)** 95–106
[18] Kalinin A V and Chekmarev D T 2014 *Mesh Methods for Boundary-Value Problems and Applications. Proc. 10th Int. Conf.* (Kazan: Kazan University) pp 356–360
[19] Kalinin A V and Kalinkina A A 2003 *Vestnik Nizhegorod. Gos. Univ. Mat. Model. Optim. Upravl.* **26 (1)** 21–38
[20] Kobelkov G M 1991 *Computational processes and systems* **8** ed. G I Marchuk (Moskow: Nauka) pp. 204-236.
[21] Kalinin A V and Kalinkina A A 2004 *Russian Mathematics (Izvestiya VUZ. Matematika)* **48 (3)** 23–31
[22] Girault V and Raviart P 1986 *Finite element methods for Navier-Stokes equations* (N.Y.: Springler-Verlag)
[23] Bossavit A 1998 *Computational electromagnetism. Variational formulations, complementarity, edge elements* (Boston: Academic Press)