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Attractor property for terminal manifold in control synthesis problem

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Abstract. General statement of control synthesis problem is considered. The problem is to find control as a function depending on the state space vector. After substituting to the system of ordinary differential equations, the function should provide the property of attractor to a terminal manifold. Symbolic regression methods for numerical solution of the problem are proposed.A numerical criterion of the attractor is set. Several examples of problem solution by the network operator method are presented.

1. Introduction

Control synthesis the problem is to find a control function. If we substitute the control function in the system of ordinary differential equations, we obtain a system without control and with new properties [1]. A particular solution of the system will reach to terminal state and will provide the optimal value of quality criterion. The difficulty of finding control function is that the same control function must provide the optimal values of the quality criterion for all the initial conditions from some state space domain [2]. Control function should depend on the constraints, quality criterion, and terminal manifold. Searching control function for creating of the attractor from the terminal manifold, we call the general synthesis problem. Having solved synthesis problem the terminal manifold becomes attractor that provides new physical properties of the control object.

Symbolic regression allows us to create numerical algorithms for functions search on the set of their descriptions. Symbolic regression methods have appeared recently, and they differ in coding. Genetic programming codes formulas by a string of symbols [3-8]. Grammatical evolution codes mathematical expressions in Backus-Naur form [9-12]. Analytical programming presents formulas in the sequence of integer numbers [13-16]. For control problems, it is proposed to use the network operator method [17]. The method applies the principle of small variations of the basic solution [18] and code expressions in integer matrices. The network operator method searches for the best solution to a set of small variations of the basic solution. There are many other methods of symbolic regression, and all of them describe complex functions as a superposition of elementary functions. We study properties of superposition and define properties of complex functions. It is important to find out what basic functions are necessary to get certain properties of the complex function.

To solve the control synthesis problem we use a symbolic regression method. For this purpose, we replace the domain of initial conditions by the set of points. For terminal manifold of non-zero dimension, we set the requirement that the particular solutions of differential equations should reach several different points of the terminal manifold.

We used the network operator method to solve the general synthesis control problem for a nonlinear system and different terminal manifolds.

2. Theory of superposition

Consider a set of functions with no more than M arguments:

$$\mathrm{F}=\{f_{1,1}(z),\ldots,f_{1,n_1}(z),f_{2,1}(z_1,z_2),\ldots,f_{2,n_2}(z_1,z_2),\ldots \\ \ldots,f_{M,1}(z_1,\ldots,z_M),\ldots,f_{M,n_M}(z_1,\ldots,z_M)\}. \tag{1}$$

To construct a mathematical expression from elementary functions (1) we should poses together with the set (1) and a set of arguments or functions without arguments:

$$\mathrm{F}_0=\{q_1,\ldots,q_{M_0}\} \tag{2}$$

Define a rule of description for superposition. We write a mathematical expression with elements of sets (1) and (2). Then we delete all brackets and insert operator icon "$\circ$" between elements. For example:

$$f_{2,j_1}(f_{1,j_2}(\ldots f_{1,j_k}(q_\alpha),f_{1,j_{k+1}}(\ldots(f_{1,j_{k+l}}(q_\beta\underbrace{)\ldots)}_{k+l}= \\ =f_{2,j_1}\circ f_{1,j_2}\circ\ldots f_{1,j_k}\circ q_\alpha\circ f_{1,j_{k+1}}\circ\ldots f_{1,j_{k+l}}\circ q_\beta. \tag{3}$$

We call sets (1), (2) as a set of basic functions.

Definition 1. Sets F and F_0 have reach ability property if for any given limited nonzero number a, $|a|<\infty$, $a\neq 0$ and small positive value ε set F_0 includes a nonzero element $q\neq 0$ and we can make superposition from F as:

$$|a-A_1\circ\ldots\circ A_l\circ q|<\varepsilon, \tag{4}$$

where $A_i\in \mathrm{F}\cup\{q\}$, $i=1,\ldots,l$.

Definition 2. Sets (1), (2) have smooth property if for given real polynomial of power n $a\neq 0$ and small positive value ε set $\mathrm{F}_0=\{q,x\}$, $q\neq 0$, x is argument of polynomial, and we can make superposition from F as:

$$|x^n+a_{n-1}x^{n-1}\ldots+a_1x+a_0-A_2\circ\ldots\circ A_l\circ x|<\varepsilon. \tag{5}$$

Definition 3. Sets (1), (2) have piecewise continuous property if for any piecewise continuous function $g(x)\in KC$ given in interval $[x_0,x_n)$ with points of discontinuity $x_1,\ldots,x_{n-1}$ and it is presented in the form of polynomial between points discontinuity:

$$r_i(x)=\sum_{k_i=0}^{m_i}a_{k_i,i}x^{k_i},\ x\in[x_i,x_{i+1}),\ i=0,\ldots,n-1, \tag{6}$$

and small positive value ε set $\mathrm{F}_0=\{q,x\}$, $q\neq 0$, x is an argument of polynomial, and we can make superposition from F as:

$$|r(x)-A_1\circ\ldots\circ A_l\circ x|<\varepsilon,\ \forall i\in\{0,\ldots,n-1\}, \tag{7}$$

where $A_i\in \mathrm{F}\cup \mathrm{F}_0$, $i=1,\ldots,l$,

$$r(x)=\begin{cases} r_0(x), \text{ if } & x\in[x_0,x_1) \\ \vdots & \\ r_{n-1}(x), \text{ if } & x\in[x_{n-1},x_n) \end{cases} \tag{8}$$

We can easy show that the set of functions:

$$\mathrm{F} = \{ f_{1,1}(z) = z,\ f_{1,2}(z) = z/2,\ f_{1,3}(z) = 2z,\ f_{1,4}(z) = -z,\ f_{1,5}(z) = \vartheta(z), \\ f_{2,1}(z_1, z_2) = z_1 + z_2,\ f_{2,2}(z_1, z_2) = z_1 z_2 \}, \tag{9}$$

where $\vartheta(z)$ is Heaviside function:

$$\vartheta(z) = \begin{cases} -1, \text{ if } z < 1 \\ 1\ , \text{ otherwise} \end{cases}, \tag{10}$$

have reachability, smooth, and piecewise continuous properties.

3. Problem statement

Given mathematical model of control object as a system of ordinary differential equations:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}), \tag{11}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathrm{U} \subseteq \mathbb{R}^m, m \le n$, U is a compact limited set, $\mathbf{x} = [x_1 \ \dots \ x_n]^T$, $\mathbf{f}(\mathbf{x}, \mathbf{u}) = [f_1(\mathbf{x}, \mathbf{u}) \ \dots \ f_n(\mathbf{x}, \mathbf{u})]^T$, $\mathbf{u} = [u_1 \ \dots \ u_m]^T$.

Given domain of initial conditions:

$$\mathrm{X}_0 \subseteq \mathbb{R}^n, \tag{12}$$

where $\forall \mathbf{x} \in \mathrm{X}_0$, $\|\mathbf{x}\| \le \alpha < \infty$.

Given terminal manifold of *n-r* dimension:

$$\varphi_i(\mathbf{x}) = 0,\ i = 1, \dots, r, \tag{13}$$

where $r \le n$. If $r = n$ we have a manifold of zero dimension or a point in the space $\mathbb{R}^n$

Given quality criterion:

$$J = \int_0^{t_f} f_0(\mathbf{x}, \mathbf{u}) dt, \tag{14}$$

where $t_f = t$, if $t \le t^+$ and $\|\varphi(\mathbf{x}(t))\| < \varepsilon$, $t^+ \le \beta < \infty$, $\varphi(\mathbf{x}) = [\varphi_1(\mathbf{x}) \ \dots \ \varphi_r(\mathbf{x})]^T$, ε is a small positive value.

It's necessary to find control function in the form:

$$\mathbf{u} = \mathbf{h}(\mathbf{x}), \tag{15}$$

where $\mathbf{h}(\mathbf{x}): \mathbb{R}^n \to \mathbb{R}^m$, $\mathbf{h}(\mathbf{x}) \in \mathrm{U}$, $\forall \mathbf{x} \in \mathbb{R}^n$.

For any initial conditions $\forall \mathbf{x}(0) = \mathbf{x}^0 \in \mathrm{X}_0$ particular solution $\mathbf{x}(t, \mathbf{x}^0)$ of differential equations:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{h}(\mathbf{x})) \tag{16}$$

for the time $t_f \le t^+$ reaches the terminal manifold (13) $\|\varphi(\mathbf{x}(t_f, \mathbf{x}^0))\| < \varepsilon$ and provides the minimal value of quality criterion (14):

$$\int_0^{t_f} f_0(\mathbf{x}(t, \mathbf{x}^0), \mathbf{h}(\mathbf{x}(t, \mathbf{x}^0))) dt = \min_{\tilde{\mathbf{u}}(\cdot) \in \mathrm{U}} \int_0^{\tilde{t}_f} f_0(\tilde{\mathbf{x}}(t, \mathbf{x}^0), \tilde{\mathbf{u}}(t)) dt, \tag{17}$$

where $\tilde{\mathbf{u}}(\cdot)$ is any admissible control, $\tilde{\mathbf{u}}(\cdot)=\mathbf{u}(t)$, $t\in[0;\tilde{t}_f]$, which provides for a particular solution $\tilde{\mathbf{x}}(t,\mathbf{x}^0)$ of differential equations $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x},\tilde{\mathbf{u}}(t))$ from the same initial condition $\forall\mathbf{x}(0)=\mathbf{x}^0\in \mathrm{X}_0$ for a limited time $\tilde{t}_f\le t^+$ to reach the terminal manifold (13), $\|\varphi(\tilde{\mathbf{x}}(t_f,\mathbf{x}^0))\|<\varepsilon$.

Equations (11) - (17) correspond to the synthesis problem of control. To obtain general synthesis problem of control, we should find such control function to make terminal manifold (13) as an attractor. If $\exists t_f\le t^+$ and $\|\varphi(\mathbf{x}(t_f,\mathbf{x}^0))\|<\varepsilon$, then $\|\varphi(\mathbf{x}(t,\mathbf{x}^0))\|<\varepsilon$, $\forall t>t_f$, and if in (13) $r<n$, $\forall t'>t_f$, $\exists\delta>0$ and $\Delta t>0$ that:

$$\|\varphi(\mathbf{x}(t',\mathbf{x}^0))-\varphi(\mathbf{x}(t'+\Delta t,\mathbf{x}^0))\|>\delta. \tag{18}$$

The main difficulty solving general synthesis problem of control (11) - (18) is a domain (12) of the initial conditions. The problem is to check the possible solution of the problem for all initial conditions from the domain.

We replace the domain of initial conditions by the set of points of initial conditions:

$$\overline{\mathrm{X}}_0=\{\mathbf{x}^{0,1},\ldots,\mathbf{x}^{0,K}\}, \tag{19}$$

where $\mathbf{x}^{0,i}\in\mathrm{X}_0\subseteq\mathbb{R}^n,\ i=1,\ldots,K$.

We change quality criterion (14):

$$\bar{J}=\sum_{i=1}^{M}\left(\int_0^{t_f} f_0(\mathbf{x},\mathbf{u})dt\right)_i, \tag{20}$$

where low index at brackets $(A)_i$ indicates that the expression in parentheses A identified with the initial conditions $\mathbf{x}^{0,i}$, $1\le i\le K$.When calculating the criterion (20)should take into account different time to reach the terminal manifold (13). In the case of not getting on of the terminal manifold (13) during the time t^+ we set $t_f=t^+$.

We define the condition of achievement of the terminal manifold (14) for the numerical synthesis as an additional criterion.We introduce a penalty for failure to comply with conditions (18). As a result, we obtain the following quality criteria:

$$\bar{J}_1=\sum_{i=1}^{M}\left(\int_0^{t_f} f_0(\mathbf{x},\mathbf{u})dt\right)_i+D\alpha_1\to\min, \tag{21}$$

$$\bar{J}_2=\sum_{i=1}^{M}\left(\|\varphi(\mathbf{x}(t,\mathbf{x}^{0,i}))\|\right)_i+D\alpha_2\to\min, \tag{22}$$

where

$$t_f=\begin{cases}t, \text{ if } t\le t^+ \text{ and } \|\varphi(\mathbf{x}(t))\|<\varepsilon\\ t^+, \text{ otherwise}\end{cases}, \tag{23}$$

$$D=\begin{cases}L-|\mathrm{G}|, \text{ if } L>|\mathrm{G}|\\ 0, \text{ otherwise}\end{cases}, \tag{24}$$

where L is a predetermined number of different points on the terminal manifold, α_1, α_2 are positive penalty values, $\mathrm{G}=\{t_1,\ldots,t_d\}$, $d\le L$, $\forall t_p\in\mathrm{G}$, and $\forall t_q\in\mathrm{G}$ conditions are fulfilled $t_p\ge t_f$,

$t_q \ge t_f$, $t_p \ne t_q$, $\|\varphi(\mathbf{x}(t_p,\mathbf{x}^{0,i}))\| < \varepsilon$, $\|\varphi(\mathbf{x}(t_q,\mathbf{x}^{0,i}))\| < \varepsilon$, and $\|\varphi(\mathbf{x}(t_p,\mathbf{x}^{0,i})) - \varphi(\mathbf{x}(t_q,\mathbf{x}^{0,i}))\| > \delta$.
From here if $t_f < t^+$, then $t_1 = t_f$, and if $t_f = t^+$, then $\mathrm{G} = \varnothing$, $D = L$.

4. Examples

Consider system of nonlinear equations with two controls:

$$\dot{x}_1 = x_2 + u_1,$$
$$\dot{x}_2 = -x_1 - x_1^3 + u_2,$$

where $-1 \le u_i \le 1$, $i = 1,2$.

Given initial conditions:

$$\bar{\mathrm{X}}_0 = \{[-1.1\ -1.1]^T, [-0.4\ -1.1]^T, [0.3\ -1.1]^T, [1\ -1.1]^T, [-1.1\ -0.4]^T,$$
$$[-0.4\ -0.4]^T, [0.3\ -0.4]^T, [1\ -0.4]^T, [-1.1\ 0.3]^T, [-0.4\ 0.3]^T, [0.3\ 0.3]^T,$$
$$[1\ 0.3]^T, [-1.1\ 1]^T, [-0.4\ 1]^T, [0.3\ 1]^T, [1\ 1]^T\}.$$

4.1. Example 1

Given a terminal manifold:

$$x_2 - 2x_1 = 0.$$

Defined functionals:

$$\bar{J}_1 = D\alpha \to \min,$$
$$\bar{J}_2 = |x_2(t^+) - 2x_1(t^+)| + D\alpha \to \min,$$

where $t^+ = 8$ s., $\alpha = 0.25$, D is defined from (24) with $\varepsilon = 0.01$, $\delta = 0.25$, $L = 4$.

We have obtained the following control function by the network operator method

$$u_i = \begin{cases} \mathrm{sgn}(u_i), \text{ if } |u_i| > 1 \\ \tilde{u}_i, \text{ otherwise} \end{cases}, \ i = 1,2, \tag{25}$$

where

$$\begin{aligned} \tilde{u}_1 &= -q_1 x_1 + \mathrm{sgn}(-x_1)\ln(|q_1 x_1| + 1) - q_2 x_2 + \mathrm{sgn}(x_2)(\exp|x_2| - 1), \\ \tilde{u}_2 &= \mathrm{sgn}(\tilde{u}_1)\ln(|\tilde{u}_1| + 1), \end{aligned} \tag{26}$$

$q_1 = 2.425354$, $q_2 = 1.000305$.

Plots of particular solutions for extreme initial conditions $\mathbf{x}^{0,1} = [-1.1\ -1.1]^T$, $\mathbf{x}^{0,4} = [1\ -1.1]^T$, $\mathbf{x}^{0,13} = [-1.1\ 1]^T$, $\mathbf{x}^{0,16} = [1\ 1]^T$ are represented in Fig. 1

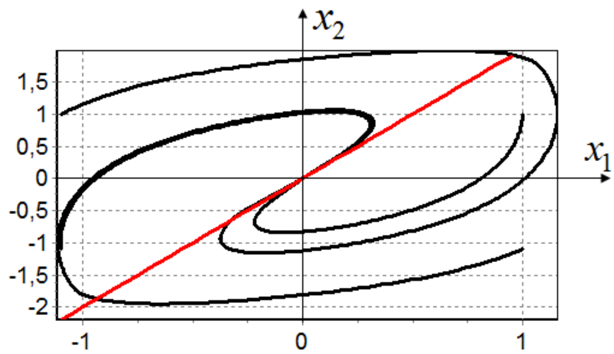

Figure 1. Solutions of the system with control (25), (26).

4.2. Example 2

Consider a terminal manifold:

$$x_1^2 + x_2^2 - 0.25 = 0 .$$

We use functionals:

$$\bar{J}_1 = \int_0^{t^+} | x_1^2(t) + x_2^2(t) - 0.25 | dt + D\alpha \to \min,$$
$$\bar{J}_2 = | x_1^2(t^+) + x_2^2(t^+) - 0.25 | + D\alpha \to \min,$$

and values of parameters are taken from example 1.

We have obtained the following control for equation (25):

$$\tilde{u}_1 = \frac{\exp(q_2 - A) - \exp(q_2)}{1 + \exp(-A)} (\mu(2B^3) + 8B^9),$$
$$\tilde{u}_2 = \vartheta\left(\frac{\exp(q_2) - \exp(q_2 - A)}{1 + \exp(-A)}\right) + \tilde{u}_1 - \tilde{u}_1^3 + \mu(2B^3) + 8B^9, \tag{27}$$

where $q_1 = 2.221802$, $q_2 = 0.370178$,

$$A = \sqrt[3]{\mu(-q_2 x_2)} + \frac{1 - \exp(q_2 x_2)}{1 + \exp(q_2 x_2)} + \ln | x_2 |,$$

$$B = C \frac{\exp(q_2) - \exp(q_2 - A)}{1 + \exp(-A)}$$

$$C = \frac{1 - \exp(-F)}{1 + \exp(-F)} \operatorname{sgn}(-q_1 x_1 + (q_1 x_1)^3) \exp(| -q_1 x_1 + (q_1 x_1)^3 | - 1),$$

$$F = \frac{1 - \exp(q_1 x_1 - (q_1 x_1)^3)}{1 + \exp(q_1 x_1 - (q_1 x_1)^3)} \exp(| q_1 x_1 | - 1)$$

$$\mu(z) = \begin{cases} \operatorname{sgn}(z), & \text{if } |z| > 1 \\ z, & \text{otherwise} \end{cases},$$

$\vartheta(z)$ is a Heaviside function(10).

Plots in Fig. 2 show solutions of the system with control (25), (27) for extreme initial conditions.

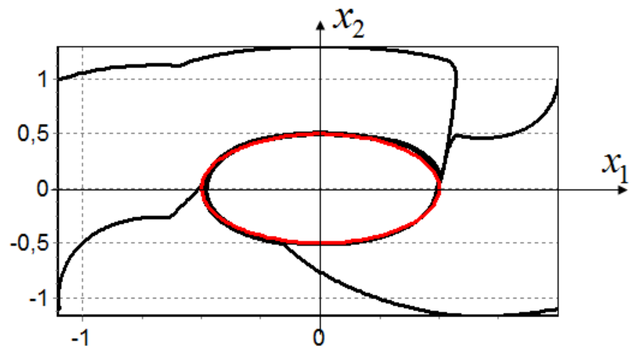

Figure 2. Solutions of system with control (25), (27).

4.3. Example 3

Consider terminal manifold:

$$-x_1+x_1^3+x_2=0$$

and functionals:

$$\bar{J}_1=\int_0^{t^+}|-x_1(t)+x_1^3(t)+x_2(t)|dt+D\alpha\to\min,$$
$$\bar{J}_1=\int_0^{t^+}|-x_1(t)+x_1^3(t)+x_2(t)|dt+D\alpha\to\min.$$

For equation (25) we have the following control:

$$\begin{aligned}\tilde{u}_1&=\mu(G)+\sqrt[3]{C}-F+\operatorname{sgn}(C)(\exp(|C|)-1),\\ \tilde{u}_2&=\operatorname{sgn}(\tilde{u}_1)\ln(|\tilde{u}_1|+1)-A+G-G^3-B^3,\end{aligned}\tag{28}$$

where $A=\mu(\operatorname{sgn}(-x_1)\ln(|-q_1x_1|+1))$, $B=\sqrt[3]{\mu(-q_2x_2)+q_2x_2}+\operatorname{sgn}(x_2)\ln(|x_2|+1)$, $G=\sqrt[3]{F}+\sqrt[3]{C}-B+A^3$, $F=\operatorname{sgn}(C)\ln(|C|+1)+B^3$, $q_1=2.105469$, $q_2=1.405334$.

Fig. 3 shows particular solutions of the system with control (25), (28) for extreme initial conditions.

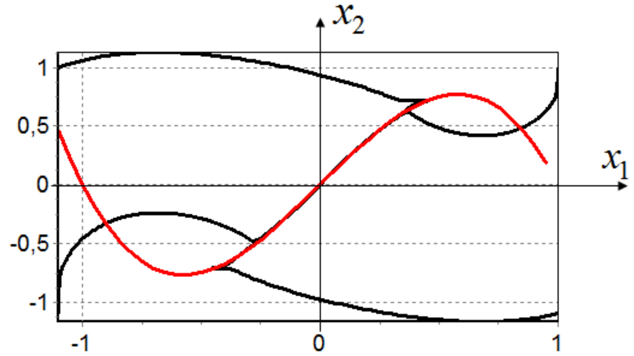

Figure 3. Solutions of the system with control (25), (28) for extreme initial conditions.

4.4. Example 4

Consider terminal manifold:

$$-x_1+x_2-x_2^3=0$$

and functionals:

$$\bar{J}_1=\int_0^{t^+}|x_1(t)+x_2(t)-x_2^3(t)|dt+D\alpha\to\min,$$
$$\bar{J}_2=|-x_1(t^+)+x_2(t^+)-x_2^3(t^+)|+D\alpha\to\min.$$

The network operator method found the following solution:

$$\begin{aligned}\tilde{u}_1&=A+B+\mu(\operatorname{sgn}(x_2)(\exp|q_2x_2|-1))+x_2+\frac{1-\exp(-x_2)}{1+\exp(-x_2)},\\ \tilde{u}_2&=\frac{1-\exp(\tilde{u}_1)}{1+\exp(\tilde{u}_1)}+\tilde{u}_1-\tilde{u}_1^3,\end{aligned}\tag{29}$$

where $q_1=2.556152$, $q_1=0.319153$, $A=\mu(1+q_1x_1-(q_1x_1)^3+\operatorname{sgn}(x_1)\sqrt{|x_1|})-q_1x_1$,

$B = C - C^3 + (1 - \exp(-x_2))(1 + \exp(-x_2))^{-1}$, $C = \mathrm{sgn}(x_2)(\exp|q_2 x_2| - 1)$.

Plots in Fig. 4 show results of simulation of the system with control (25), (29) for extreme initial conditions.

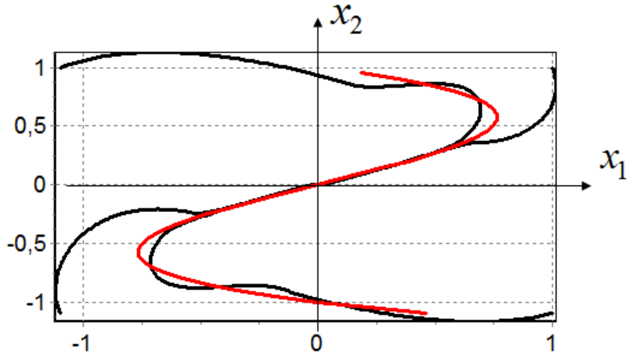

Figure 4. Solutions of the system with control (25), (29) for extreme initial conditions.

5. Conclusion

We formulated the general control synthesis problem. A solution of this problem makes terminal manifold an attractor. We offered to use symbolic regression to the numerical solution of the problem and specified criteria for numerical synthesis. Symbolic regression finds mathematical expressions as a superposition of elementary functions. Some tasks of synthesis for the nonlinear system have been solved by the network operator method. All examples show how terminal manifolds become attractors by solving general control synthesis problem.

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