IOP Conf. Series: Materials Science and Engineering **148** (2016) 012002 doi:10.1088/1757-899X/148/1/012002

# Analysis of limit forces on the vehicle wheels using an algorithm of Dynamic Square Method

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**Abstract.** This article presents a method named as Dynamic Square Method (DSM) used for dynamic analysis of a vehicle equipped with a four wheel drive system. This method allows determination of maximum (limit) forces acting on the wheels. Here, the maximum longitudinal forces acting on the wheels are assumed and then used to predict whether they can be achieved by a specific dynamic motion or whether the actual friction forces under a given wheel is large enough to transfer lateral forces. For the analysis of DSM a four wheel vehicle model is used. On the basis of this characteristic it is possible to determine the maximum longitudinal force acting on the wheels of the given axle depending on the lateral acceleration of the vehicle. The results of this analysis may be useful in the development of a control algorithm used for example in active differentials.

## 1. Introduction

Analysing vehicle dynamics while cornering or while driving in a curvilinear motion, taking into account the driving forces and braking is an important issue. The analysis helps to improve active safety of the vehicle, and also in the creation of a sophisticated drive technology especially with an all-wheel drive. The Dynamic Square Method (DSM) allows us to define the maximum longitudinal force at the wheels as a function of lateral and longitudinal acceleration obtained by the vehicle while driving in a rectilinear or curvilinear motion. Knowledge of available forces (longitudinal and lateral) on wheels allows us e.g. to develop a system for the control of torque (Torque Vectoring (TV)) [5, 6].The main aim of this study was an attempt to comprehend and utilize the DSM for four-wheel vehicle model. An analysis was attempted for the change in vertical load on each wheel while driving in both a curve and on a straight track. Maximum longitudinal forces on the wheels along with lateral an longitudinal vehicle acceleration were determined as a result.

## 2. Analysis Dynamic Square Method for using four wheel vehicle model

The DSM firstly was presented in 1995 by Kato M., Isoda K. and Yuasa H. [2]. This method was mentioned by Klomp M. in his articles [3, 4]. This method allows to determine the actual maximum longitudinal and lateral forces which can be transferred by the wheels of the vehicle on the road. On the basis of the actual forces on the wheels longitudinal and lateral acceleration of the vehicle can be calculated. Practical use of DSM was presented by the author by a single track vehicle model [1]. This article presents the analysis of Dynamic Square Method for a two track vehicle model powered by a four wheel drive (figure 1 and 2).

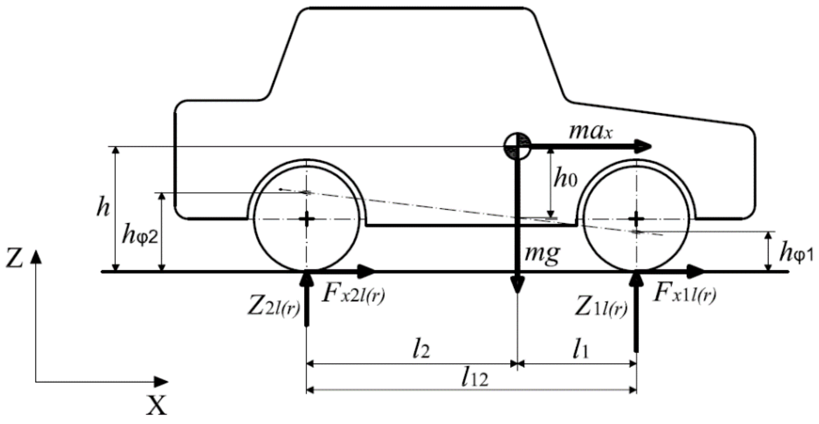


**Figure1.** Four-wheel vehicle model – side view.

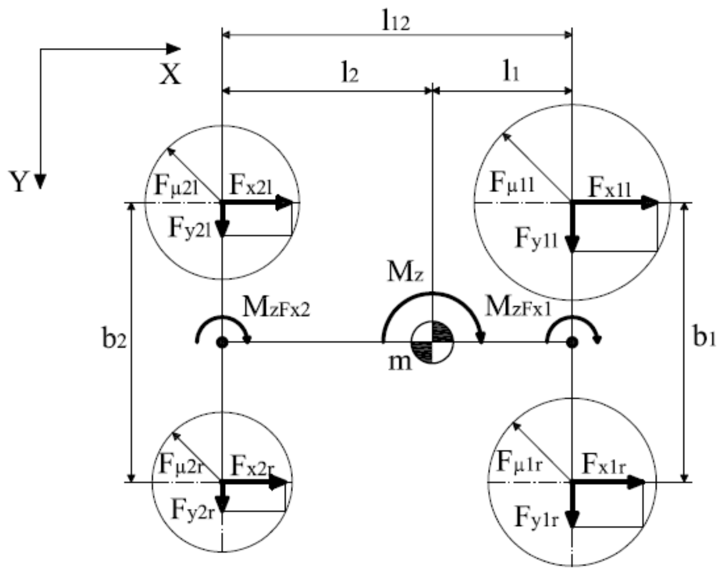


**Figure 2.** Four-wheel vehicle model – plane view.

The following assumptions were made for the analysis of a two track model by the DSM:

- a three dimensional model is used,
- vehicle curvilinear motions assumed as to the vehicle turning towards the right,
- traction forces $F_{n1(2)}$ and braking forces are transferred for all wheels to the ground (four wheel drive system),
- traction force or braking force on a given axis is the sum of the longitudinal forces on each of its wheels i.e. $F_{x1} = F_{x1l} + F_{x1r}$ and $F_{x2} = F_{x2l} + F_{x2r}$,
- a symmetrical distribution of driving and braking forces exists between the wheels of each axle, $F_{x1l} = F_{x1r}$ and $F_{x2l} = F_{x2r}$,
- maximal longitudinal acceleration $a_x$ = 9,81 m/s$^2$,
- maximal longitudinal deceleration $a_x$= 9,81 m/s$^2$,
- account resistance forces $F_p$, are neglected,
- lateral load transfer between left and right wheels of the front axle $\Delta F_{y1}$ and between left and right wheel of the rear axle $\Delta F_{y2}$ for curvilinear motion is taken into account,

- front vehicle stiffness $k_{\varphi 1}$ and rear stiffness $k_{\varphi 2}$, are taken into account for roll conditions at the front,
- longitudinal load transfer between front and rear axis $\Delta F_x$, are taken into account,
- all wheels have the same value of friction coefficient in all directions ($\mu_m$ = 1), friction coefficient of front $\mu_{m1}$ and rear $\mu_{m2}$ wheels is the same and equals $\mu_m$.

When considering issues of classic dynamics it will possible to calculate forces on the wheels (longitudinal, lateral) knowing the acceleration (longitudinal, lateral) or velocity (longitudinal, lateral) of the vehicle.

On the basis of the assumed longitudinal forces on the wheels $F_x$, longitudinal acceleration $a_x$ of the vehicle is calculated, as the quotient of the sum of the longitudinal forces acting on the wheels of the vehicle $\Sigma F_x$ and its mass m.

$$a_x = \frac{\Sigma F_x}{m} = \frac{F_{x1} + F_{x2}}{m} = \frac{F_{x1l} + F_{x1r} + F_{x2l} + F_{x2r}}{m} \tag{1}$$

where:

$F_{x1l}$ – longitudinal force on the left front wheel (during acceleration $F_{x1l} = F_{n1l}$, during deceleration $F_{x1l} = F_{h1l}$),
$F_{x1r}$ – longitudinal force on the right front wheel (during acceleration $F_{x1r} = F_{n1r}$, during deceleration $F_{x1r} = F_{h1r}$),
$F_{x2l}$ – longitudinal force on the left rear wheel (during acceleration $F_{x2l} = F_{n2l}$, during deceleration $F_{x2l} = F_{h2l}$),
$F_{x2r}$ – longitudinal force on the right rear wheel (during acceleration $F_{x2r} = F_{n2r}$, during deceleration $F_{x2r} = F_{h2r}$).

During acceleration or deceleration lateral weight transfer $\Delta F_x$ is changed.

$$\Delta F_x = \frac{m a_x \boldsymbol{h}}{l_{12}} \tag{2}$$

where:
$h$ – height of gravitational centre [m],
$l_{12}$ – wheel base [m].

Lateral load transfer at the front axle $\Delta F_{y1}$ and $\Delta F_{y2}$ at rear axle are calculated while the vehicle drives along a curvilinear path.
During curvilinear motion of the vehicle lateral load transfer front axle $\Delta F_{y1}$ and rear axle $\Delta F_{y2}$ are changing also.

$$\Delta F_{y1} = \frac{k_{\varphi 1} m a_y \boldsymbol{h_0}}{b_1(k_{\varphi 1} + k_{\varphi 2} - mg\boldsymbol{h_0})} + \frac{m a_y l_2 \boldsymbol{h}_{\varphi 1}}{l_{12} b_1} \tag{3}$$

$$\Delta F_{y2} = \frac{k_{\varphi 2} m a_y \boldsymbol{h_0}}{b_2(k_{\varphi 1} + k_{\varphi 2} - mg\boldsymbol{h_0})} + \frac{m a_y l_1 \boldsymbol{h}_{\varphi 2}}{l_{12} b_2} \tag{4}$$

where:
$m$ – vehicle mass [kg],
$a_y$ – vehicle lateral acceleration [m/s$^2$],
$k_{\varphi 1}$ – front vehicle roll stiffness [Nm/rad],

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$k_{\varphi 2}$ – rear vehicle roll stiffness [Nm/rad],
$h_{\varphi 1}$ – height of front roll centre [m],
$h_{\varphi 2}$ – height of rear roll centre [m],

$h_0$ – distance between gravitational centre and roll axis [m], $\boldsymbol{h_0} = h - \frac{\boldsymbol{h_{\varphi 1}} - \boldsymbol{h_{\varphi 2}}}{l_{12}} l_1 + \boldsymbol{h_{\varphi 1}}$
$\boldsymbol{h_0} = h - \frac{\boldsymbol{h_{\varphi 1}} - \boldsymbol{h_{\varphi 2}}}{l_{12}} l_1 + \boldsymbol{h_{\varphi 1}}$,
$h$ – height of gravitational centre[m],
$b_1$ –front track [m],
$b_2$ – rear track [m],
$g$ – acceleration of gravity [m/s$^2$],
$l_1$ –distance between front axle and gravitational centre [m],
$l_2$ – distance between rear axle and gravitational centre [m],
$l_{12}$ – wheel base [m].

To calculate the change of lateral load transfer in four-wheeled vehicle model (equations (2.3) and (2.4)) knowledge of the lateral acceleration is required. For this purpose a previously made two-wheel vehicle model [1] and on the basis of determined lateral acceleration $a_y$.
Knowledge of lateral load transfer between left and right wheels of the front axle $\Delta F_{y1}$and lateral load transfer between left and right wheels of the rear axle $\Delta F_{y2}$ and longitudinal load transfer between front and rear axis $\Delta f_x$ makes it possible to calculate vertical loads of each wheel $Z_{1(2)l(r)}$.Signs of longitudinal and lateral load transfer depends on whether the wheel is loaded or unloaded while driving.

$$Z_{1l} = Z_{1lstat} - \frac{\Delta F_x}{2} + \Delta F_{y1} = \frac{m_1}{2} g - \frac{\Delta F_x}{2} + \Delta F_{y1} \tag{5}$$

$$Z_{1r} = Z_{1rstat} - \frac{\Delta F_x}{2} - \Delta F_{y1} = \frac{m_1}{2} g - \frac{\Delta F_x}{2} - \Delta F_{y1} \tag{6}$$

$$Z_{2l} = Z_{2lstat} + \frac{\Delta F_x}{2} + \Delta F_{y2} = \frac{m_2}{2} g + \frac{\Delta F_x}{2} + \Delta F_{y2} \tag{7}$$

$$Z_{2r} = Z_{2rstat} + \frac{\Delta F_x}{2} - \Delta F_{y2} = \frac{m_2}{2} g + \frac{\Delta F_x}{2} - \Delta F_{y2} \tag{8}$$

where:
$m_1$ – front mass [kg],
$m_2$ – rear mass [kg],
$Z_{1l(r)stat}$– static vertical load of left (right) of front vehicle wheel [N],
$Z_{2l(r)stat}$– static vertical load of left (right) of rear vehicle wheel [N].

Calculated vertical loads $Z_{1(2)l(r)}$allows to determine the friction forces under each wheel $F_\mu$.

$$F_{\mu 1l} = \mu_{m1l} Z_{1l} \tag{9}$$

$$F_{\mu 1r} = \mu_{m1r} Z_{1r} \tag{10}$$

$$F_{\mu 2l} = \mu_{m2l} Z_{2l} \tag{11}$$

$$F_{\mu 2r} = \mu_{m2r} Z_{2r} \tag{12}$$

where:

$\mu_{m1l(r)}$– friction coefficient of left (right) front wheel,
$\mu_{m2l(r)}$ – friction coefficient of left (right) rear wheel.

An initial assumption that the longitudinal force $F_x$ on the wheels cannot be greater than the friction force on the given wheel $F_\mu$. This situation is represented in figure 3 by the tire friction circle [7].

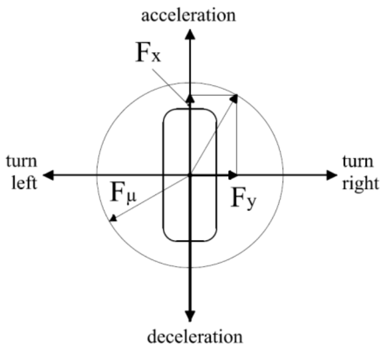


**Figure 3.** Tire friction circle

The figure 3 depicts tire friction circle which illustrates relationships between longitudinal forces $F_x$ (driving $F_n$ and braking $F_h$ ), longitudinal forces $F_y$ and friction forces $F_\mu$.
As it was assumed earlier that the longitudinal force on any wheel $F_x$is smaller than the friction forces on the wheel $F_\mu$ it will be possible to transfer on the road the lateral force $F_y$:

$$F_{y1l} = \left(F_{\mu 1l}^2 - F_{x1l}^2\right)^{\frac{1}{2}} \tag{13}$$

$$F_{y1r} = \left(F_{\mu 1r}^2 - F_{x1r}^2\right)^{\frac{1}{2}} \tag{14}$$

$$F_{y2l} = \left(F_{\mu 2l}^2 - F_{x2l}^2\right)^{\frac{1}{2}} \tag{15}$$

$$F_{y2p} = \left(F_{\mu 2r}^2 - F_{x2r}^2\right)^{\frac{1}{2}} \tag{16}$$

It can be calculated as the total lateral force which can be transferred along a given axis.

$$F_{y1} = F_{y1l} + F_{y1r} \tag{17}$$

$$F_{y2} = F_{y2l} + F_{y2r} \tag{18}$$

The longitudinal and lateral forces acting in the tire-road contact generate yaw moment $M_z$ about the vertical axis of vehicle $Z$.
Yaw moment $M_{zFy}$ depends on the lateral forces $F_y$, where $F_{y1}$ is lateral force at the front and $F_{y2}$ at the rear.

Yaw moment $M_{zFy}$ acting around the vertical axis of vehicle $Z$ is caused by longitudinal forces at the front axis $F_{y1}$ and rear axis $F_{y2}$ respectively, and are expressed by the following equations:

$$M_{zFy} = F_{y1} l_1 - F_{y2} l_2 \tag{19}$$

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The difference in the friction coefficient between left and right side i.e. $\mu_{m1(2)l}$ and $\mu_{m1(2)r}$ respectively gives rise to a yaw moment $M_{zFx}$ which leads to a difference in lateral load transfer between left and right wheels of the front axle $\Delta F_{y1}$ and lateral load transfer between left and right wheels of the rear axle $\Delta F_{y2}$. The difference between longitudinal forces on the wheels of the front axle $\Delta F_{y1}$ and the rear axle is depicted below.

$$\Delta F_{x1} = \left(F_{\mu m1l}{}^{2} - F_{y1l}{}^{2}\right)^{\frac{1}{2}} sgn(F_{x1l}) - \left(F_{\mu m1r}{}^{2} - F_{y1r}{}^{2}\right)^{\frac{1}{2}} sgn(F_{x1r}) \tag{20}$$

$$\Delta F_{x2} = \left(F_{\mu m2l}{}^{2} - F_{y2l}{}^{2}\right)^{\frac{1}{2}} sgn(F_{x2l}) - \left(F_{\mu m2r}{}^{2} - F_{y2r}{}^{2}\right)^{\frac{1}{2}} sgn(F_{x2r}) \tag{21}$$

Orientation of the sign for the difference between longitudinal forces on the front axis $\Delta F_{x1}$ and rear axis $\Delta F_{x2}$ depends on the direction of longitudinal forces $F_x$ on the given wheel.
Yaw moment $M_{zFx1(2)}$ acting around the vertical axis of vehicle $Z$ is caused by the difference in longitudinal forces at the front axis $\Delta F_{x1}$ and rear axis $\Delta F_x$ respectively, and are expressed by the following equations:

$$M_{zFx1} = \Delta F_{x1lr} \frac{b_1}{2} \tag{22}$$

$$M_{zFx2} = \Delta F_{x2lr} \frac{b_2}{2} \tag{23}$$

Total yaw moment $M_{zFx}$ caused by the difference of longitudinal forces on the wheels of each axles accepts form:

$$M_{zFx} = M_{zFx1} + M_{zFx2} \tag{24}$$

Total yaw moment $M_z$ around the vehicle axis $Z$ is a superposition of yaw moments which come from lateral forces $M_{zFy}$ and yaw moment resulting from the difference between longitudinal forces on the wheels of one axle $M_{zFx}$.

$$M_z = M_{zFy} + M_{zFx} \tag{25}$$

Total yaw moment $M_z$ allows us to define steering characteristics. If moment equals zero($M_z$= 0),then vehicle has neutrally steered. Whereas, if $M_z$< 0 then vehicle is under steered and when $M_z$> 0 vehicle is over steered.
Previously calculated yaw moment $M_z$ will be required to calculate the lateral acceleration $a_y$ for four-wheel vehicle model represented by the following equations:

$$\begin{cases} a_y = \dfrac{F_{y1} + F_{y2}}{m}, & if\, M_z = 0 \\ a_y = \dfrac{F_{y1} + \frac{M_{zFx}}{l_{12}}}{m_1}, & if\, M_z < 0 \\ a_y = \dfrac{F_{y2} - \frac{M_{zFx}}{l_{12}}}{m_2}, & if\, M_z > 0 \end{cases} \tag{26}$$

The process is repeated several times (n) for calculating longitudinal forces which are a function of longitudinal and lateral acceleration of the vehicle. On the basis of the above shown scheme the algorithm is created. This was done using numerical calculation by the program Scilab. The number of iterations were 200 and as a result the lateral acceleration of the vehicle is calculated. This is done by joining points with similar values of lateral acceleration. The result was an iso-altitude lateral acceleration.10 isolines can be seen for greater readability of the graph.
Next, a principle of operation of DSM will be presented by the example of a car model, whose data is presented by Table 1. and 2.

**Table1.**Vehicle data

| | |
|---|---|
| Vehicle mass $m$ [kg] | 1450 |
| Front mass$m_1$ [kg] | 870 |
| Rear mass$m_2$ [kg] | 580 |
| Friction coefficient of left side of vehicle $\mu_{ml}$ [-] | 1,0 |
| Friction coefficient of right side of vehicle$\mu_{mr}$ [-] | 1,0 |
| Distance between front axle and gravitational center $l_1$ [m] | 1,06 |
| Distance between rear axle and gravitational center $l_2$ [m] | 1,59 |
| Wheel base $l_{12}$ [m] | 2,65 |
| Front track$b_1$ [m] | 1,50 |
| Rear track$b_2$ [m] | 1,50 |
| Height of gravitational $h$ [m] | 0,53 |
| Height of front roll center $h_{\varphi 1}$ [m] | 0,10 |
| Height of front roll center $h_{\varphi 2}$ [m] | 0,15 |
| Distance between t center and roll axis $h_0$ [m] | 0,41 |
| Front vehicle roll stiffness $k_{\varphi 1}$ [Nm/rad] | 50 000 |
| Rear vehicle roll stiffness $k_{\varphi 2}$ [Nm/rad] | 30 000 |

**Table2.**Assumed driving and braking forces for the reference vehicle

| | |
|---|---|
| Driving force on left front wheel $F_{n1l}$ [N] | (0; 6000) |
| Driving force on right front wheel $F_{n1r}$ [N] | (0; 6000) |
| Driving force on left rear wheel $F_{n2l}$ [N] | (0; 6000) |
| Driving force on right rear wheel $F_{n2r}$ [N] | (0;6000) |
| Braking force on left front wheel $F_{h1l}$ [N] | (- 6000; 0) |
| Braking force on right front wheel $F_{h1r}$ [N] | (- 6000; 0) |
| Braking force on left rear wheel $F_{h2l}$[N] | (- 3500; 0) |
| Braking force on right rear wheel $F_{h2r}$ [N] | (- 3500; 0) |
| Driving force on front axle $F_{n1}$ [N] | (0; 12 000) |
| Driving force on rear axle $F_{n2}$ [N] | (0; 12 000) |
| Braking force on front axle $F_{h1}$ [N] | (- 12 000; 0) |
| Braking force on rear axle $F_{h2}$ [N] | (- 7000; 0) |

Figure 4 depicts the graphical forces on the wheels of the vehicle calculated by the DSM for a four-wheel vehicle model obtained on the basis of the above calculations.

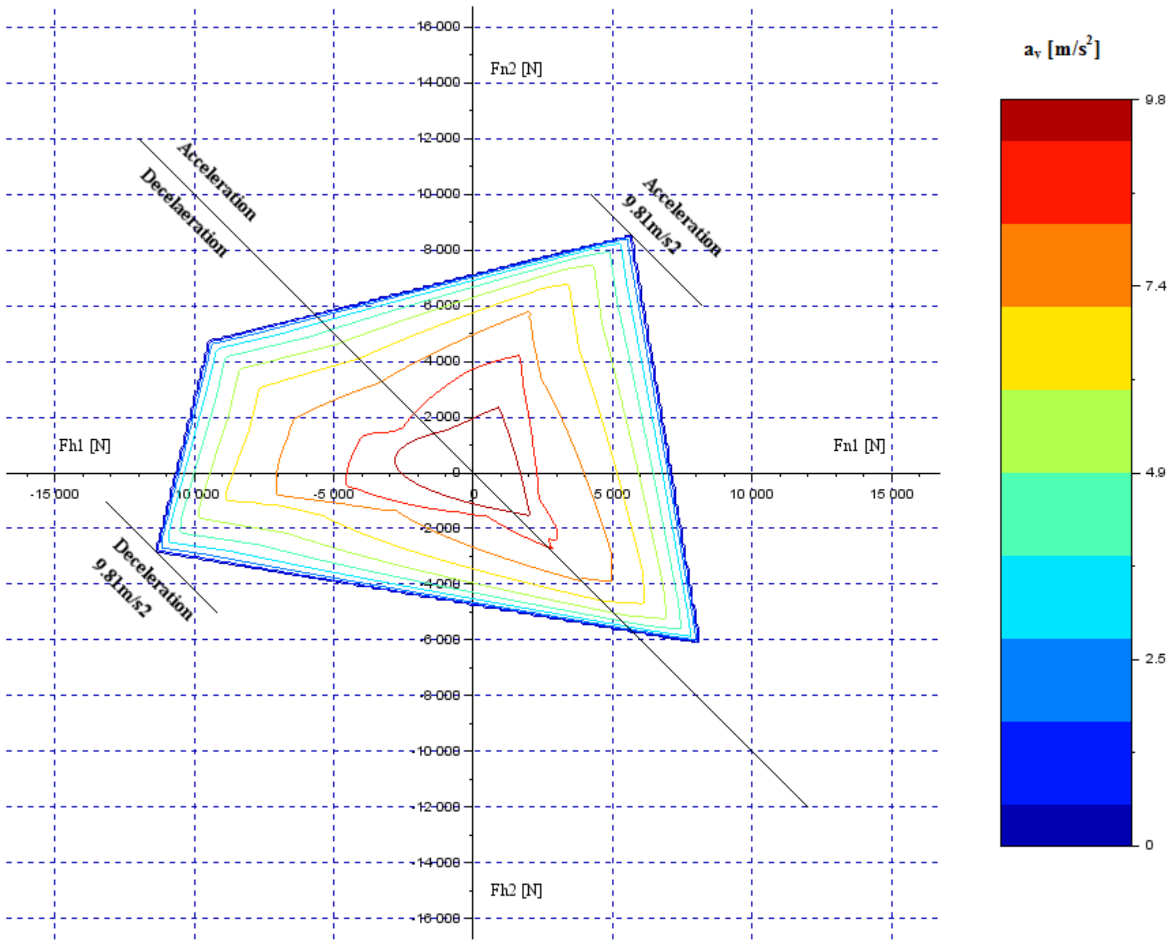


**Figure 4.** Limit forces on the vehicle wheels for using DSM for four-wheel vehicle model.

Figure 4 presents the limit forces acting on the wheels of the vehicle obtained via Dynamic Square Method for the four-wheel vehicle model. The analysis of the above characteristics results in understanding the relationship between the real longitudinal forces on the wheels of the vehicle, lateral acceleration $a_y$ and longitudinal $a_x$. Analysing Figure 2.4 one can determine the limit of traction force acting on the axis which in turn leads us to value of lateral acceleration. The longitudinal acceleration of the vehicle results from the sum of the limit forces on the wheels front and rear axle results.

Figure 4 shows the limit forces on the wheels of the vehicle obtained via Dynamic Square Method for the four-wheel vehicle model. The analysis of the above characteristics of the resulting relationship between the real longitudinal forces on the wheels of the vehicle and achieved the lateral acceleration $a_y$ and longitudinal acceleration $a_x$. Analysing Figure 2.4 can determine which limit the traction force on the axis is capable of achieving a function of the vehicle lateral acceleration. With the sum of the limit forces on the wheels front and rear axle results the longitudinal acceleration of the vehicle. So you know what is the relationship between the longitudinal and lateral acceleration of the vehicle.

It is also worth noting fact that the right upper part of the graph (for positive longitudinal forces on the wheels) and the left lower right upper part of the graph (negative longitudinal forces on the wheels) have real sense.

Also one can define the maximum driving forces on the wheels during acceleration and braking. Thus, the longitudinal forces on the wheels of the vehicle located in the upper right corner of the graph correspond to the values of the driving forces at the front and rear wheels $F_{n1(2)}$ during the rectilinear

motion when maximum longitudinal acceleration $a_x$ equals 9,81 m/s$^2$. It can be stated that the maximum driving force for the front wheels and the rear (see Table 2.2) cannot be realized. The maximum value of the driving forces obtained by the car model on the front and rear wheels during the rectilinear motion of the vehicle are as follows ($a_y$ = 0 m/s$^2$): $F_{n1} \approx$ 5700 N, $F_{n2} \approx$ 8500 N.
The values of longitudinal forces at the wheels contained in the lower left corner of the graph represent the vehicle driving along a rectilinear motion with a maximum longitudinal deceleration $a_x$ = − 9,81 m/s$^2$.In this situation, the vehicle model gives us values of braking force at the front and rear wheels$F_{h1} \approx -$ 11 400 N and$F_{h2} \approx -$ 2800 N respectively.
The inner part of the graph (Figure 2.4) consists of iso-altitude lines depicting constant value of lateral acceleration $a_y$. Diagonal lines represent the relationship between longitudinal force at the front axle$F_{x1}$and the rear axle$F_{x2}$and prove that$F_{x1}+F_{x2}=$ *constant*, which means that the sum of the longitudinal forces on the front and rear wheels is always constant for the given longitudinal acceleration $a_x$.

## 3. Conclusions

The presented algorithm Dynamic Square Method has great development potential. On the basis of DSM can be determined the real lateral and longitudinal forces acting on the vehicle wheels, as well as lateral and longitudinal acceleration resulting from the action of these forces.
There are plans for the further expansion of the model for e.g. taking into account a model describing the contact of the wheel with the road and its use in creating a control system used to calculate driving and braking force (Torque Vectoring).

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