Mathematical modelling of contact of ruled surfaces: theory and practical application

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Abstract. In the theory of ruled surfaces there are well known researches of contact of ruled surfaces along their common generator line (Klein image is often used [1]). In this paper we propose a study of contact of nondevelopable ruled surfaces via the dual vector calculus. The advantages of this method have been demonstrated by E. Study, W. Blaschke and D. N. Zeiliger in differential geometry studies of ruled surfaces in space R^3 over the algebra of dual numbers. A practical use of contact is demonstrated by the example modeling of the working surface of the progressive tool for tillage.

1. Fundamentals of theory

The equation of a ruled surface can be represented as a unit dual vector-function with real parameter t: $\boldsymbol{A}_1(t)=\boldsymbol{a}_{01}(t)+\omega\boldsymbol{a}_{11}(t), |\boldsymbol{A}_1(t)|=1, \omega^2=0$, where $\boldsymbol{a}_{01}(t)$ – a unit generator vector of the ruled surface, $\boldsymbol{a}_{11}(t)$ – a momentum vector of the generator with respect to the origin. $\boldsymbol{A}_1(t)$ is $n+1$ times continuously differentiable. It is also known that a trihedron of the ruled surface consists of dual unit vectors:

$$\boldsymbol{A}_1; \boldsymbol{A}_2=\frac{\boldsymbol{A}_1'}{|\boldsymbol{A}_1'|}; \boldsymbol{A}_3=\boldsymbol{A}_1\times\boldsymbol{A}_2,$$

where $\boldsymbol{A}'_1=d\boldsymbol{A}_1/dt$; $H_A=|\boldsymbol{A}'_1|$; $\boldsymbol{A}_2$ – a central normal vector; $\boldsymbol{A}_3$ – a central tangent vector.

And we also used an inner geometric parameter of the ruled surface - an element of a dual arc:

$$ds_A=ds_{A0}+\omega\cdot ds_{A1}=H_A dt, \omega^2=0,$$

which means a dual angle between its two infinitesimally close generators.

A distribution parameter of the ruled surface is deduced from the element of the dual arc:

$$p=\frac{ds_{A1}}{ds_{A0}}.$$

Trihedron derivation equations of W. Blaschke [2] are as follows:

$$\boldsymbol{A}_1'=H_A\cdot\boldsymbol{A}_2;\ \boldsymbol{A}_2'=-H_A\cdot\boldsymbol{A}_1+Q_A\cdot\boldsymbol{A}_3;\ \boldsymbol{A}_3'=-Q_A\cdot\boldsymbol{A}_2,$$

where $H_A=h_{A0}+\omega h_{A1}$, $Q_A=q_{A0}+\omega q_{A1}=(\boldsymbol{A}_1\boldsymbol{A}'_1\boldsymbol{A}''_1)/H_A^2$, $\omega^2=0$.

Let us consider the contact of two nondevelopable ruled surfaces parameterized by $\boldsymbol{A}_1(t)$ and $\boldsymbol{B}_1(t)$ with common ruling $\boldsymbol{a}_{01}(t)=\boldsymbol{b}_{01}(t)$, when parameter value $t=t_0$. Initially we will introduce the dual vector of divergence $\boldsymbol{G}(t)$ in their common ruling. Since each unit dual vector function $\boldsymbol{A}_1(t)$ and $\boldsymbol{B}_1(t)$ can be represented as a Taylor series it can be written as:

$$\boldsymbol{G}(t)=\boldsymbol{A}_1(t_0)-\boldsymbol{B}_1(t_0)+[\boldsymbol{A}_1'(t_0)-\boldsymbol{B}_1'(t_0)]\cdot\Delta t+[\boldsymbol{A}_1''(t_0)-\boldsymbol{B}_1''(t_0)]\cdot\frac{\Delta t^2}{2!}+.. \tag{1}$$

We will define the concept of an order of contact of two ruled surfaces in their common generator line. Each of the considered dual vector functions has the following coordinate representation:

$$\begin{gathered}\boldsymbol{A}_1(t)=\boldsymbol{i}\cdot x_A(t)+\boldsymbol{j}\cdot y_A(t)+\boldsymbol{k}\cdot z_A(t),\\ \boldsymbol{B}_1(t)=\boldsymbol{i}\cdot x_B(t)+\boldsymbol{j}\cdot y_B(t)+\boldsymbol{k}\cdot z_B(t),\end{gathered} \tag{2}$$

where x_A, y_A, z_A; x_B, y_B, z_B – dual scalar functions. It allows one to write an expression for dual vector $\boldsymbol{G}(t)$:

$$\boldsymbol{G}(t)=\boldsymbol{A}_1(t)-\boldsymbol{B}_1(t)=\boldsymbol{i}\cdot x_G+\boldsymbol{j}\cdot y_G+\boldsymbol{k}\cdot z_G=\boldsymbol{i}(x_A-x_B)+\boldsymbol{j}(y_A-y_B)+\boldsymbol{k}(z_A-z_B). \tag{3}$$

Dual vector $\boldsymbol{G}(t)$ characterizes the closeness of ruled surfaces $\boldsymbol{A}_1(t)$ and $\boldsymbol{B}_1(t)$ in infinitesimal neighborhood for their common generator at parameter $t = t_0$. It is defined by two generators $\boldsymbol{a}'_{01}(t)$ and $\boldsymbol{b}'_{01}(t)$, each of which is shifted in their ruled surface along the same dual arc $ds_A = ds_B$ relatively their common generator $\boldsymbol{a}_{01}(t) = \boldsymbol{b}_{01}(t)$. Let us introduce the notion of the order contact of two ruled surfaces in their common generator. We will estimate an order of infinitesimality of module $|\boldsymbol{G}(t)|$ relatively dual infinitesimal Δs. If n – positive integer then the following condition holds:

$$\lim_{\Delta s\to 0}\frac{|\boldsymbol{G}|}{\Delta s^n}=0, \tag{4}$$

thus, we will assume that ruled surfaces in their common generator $\boldsymbol{a}_{01}(t_0) = \boldsymbol{b}_{01}(t_0)$ are not lower than the n-th order contact. But if n - maximal then ruled surfaces are exactly in the n-th order contact.

Because of $ds=ds_0 + \omega ds_1=h_0dt + \omega h_1dt$, $\omega^2=0$, where $ds_0 =h_0dt$, $ds_1=h_1dt$, so condition $\Delta s\to 0$ can be replaced by the following two conditions: $\Delta s_0\to 0$ and $\Delta s_1\to 0$, that in case of $h_0\neq 0$ and $h_1\neq 0$ leads to $\Delta t\to 0$. Indeed ds_0 and dt, ds_1 and dt – pairs of equivalent infinitesimals. As a result we obtain the following condition for estimating the order contact of ruled surfaces in their common generator:

$$\lim_{\Delta t\to 0}\frac{|\boldsymbol{G}(t)|}{\Delta t^n}=0. \tag{5}$$

Theorem. To fulfill the condition of (5) it is necessary and sufficiently that the following equalities are carried out

$$\begin{gathered}x_G(t_0)=0,\ x'_G(t_0)=0,\ x''_G(t_0)=0,..,x_G{}^n(t_0)=0;\\ y_G(t_0)=0,\ y'_G(t_0)=0,\ y''_G(t_0)=0,..,y_G{}^n(t_0)=0;\\ z_G(t_0)=0,\ z'_G(t_0)=0,\ z''_G(t_0)=0,..,z_G{}^n(t_0)=0;\end{gathered} \tag{6}$$

Sufficiency. Assume that equalities (6) hold. Then from (1) it follows that:

$$x_G(t)=x_A(t)-x_B(t)=x^{n+1}(\delta_1)\cdot\frac{\Delta t^{n+1}}{(n+1)!};\ y_G(t)=y_A(t)-y_B(t)=y^{n+1}(\delta_2)\cdot\frac{\Delta t^{n+1}}{(n+1)!};$$

$$z_G(t)=z_A(t)-z_B(t)=z^{n+1}(\delta_3)\cdot\frac{\Delta t^{n+1}}{(n+1)!},$$

where δ_1, δ_2, δ_3 – values of parameter t, which satisfy the following conditions: $t_0<\delta_1<t$, $t_0<\delta_2<t$, $t_0<\delta_3<t$.

Thus, the condition of (5) holds.

Necessity. Let us assume that at least one of equalities (6) does not hold, for example $x_G{}^k(t_0)=x_A{}^k(t_0)-x_B{}^k(t_0)\neq 0$; $k<n$.

Thus, $x_G(t)=x_A(t)-x_B(t)=\left[x_A{}^k(t_0)-x_B{}^k(t_0)\right]\cdot\frac{\Delta t^k}{k!}+\dots,$

hence $\lim_{\Delta t\to 0}\frac{|\boldsymbol{G}(t)|}{\Delta t^n}\neq 0$, $k<n$.

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So the condition of (5) does not hold.

Let us consider the primary orders of contact of ruled surfaces.

2. The orders of contact

2.1. Order n=0 of contact of ruled surfaces

$$\boldsymbol{A}_1(t_0)=\boldsymbol{B}_1(t_0); \boldsymbol{A}_1'(t_0)\neq \boldsymbol{B}_1'(t_0). \tag{7}$$

In this case ruled surfaces are intersecting along their common generator.

2.2. Order n=1 of contact of ruled surfaces

$$\boldsymbol{A}_1(t_0)=\boldsymbol{B}_1(t_0); \boldsymbol{A}_1'(t_0)=\boldsymbol{B}_1'(t_0); \boldsymbol{A}_1''(t_0)\neq \boldsymbol{B}_1''(t_0). \tag{8}$$

Ruled surfaces are exactly in the first order contact in their common generator $\boldsymbol{a}_{01}(t_0) = \boldsymbol{b}_{01}(t_0)$.

From equalities of dual arcs $ds_A = ds_B$ it follows that $H_A dt = H_B dt$, i.e. $H_A = H_B$. Since $\mathbf{B}'_1=H_B\cdot\mathbf{B}_2$, where $H_B=\left|\boldsymbol{B}_1'\right|$, then from condition $\boldsymbol{A}'_1(t_0)=\boldsymbol{B}'_1(t_0)$ we obtain that in the common generator the following equalities hold:

$$\boldsymbol{A}_1=\boldsymbol{B}_1;\ \boldsymbol{A}_2=\boldsymbol{B}_2;\ \boldsymbol{A}_3=\boldsymbol{B}_3;\ H_A=H_B. \tag{9}$$

For order $n = 1$ of contact ruled surfaces have coincided with trihedrons ($\boldsymbol{a}_{01}$, $\boldsymbol{a}_{02}$, $\boldsymbol{a}_{03}$) and ($\boldsymbol{b}_{01}$, $\boldsymbol{b}_{02}$, $\boldsymbol{b}_{03}$) at the center point of their common generator (see Figure 1, a).

2.3. Order n=2 of contact of ruled surfaces

$$\boldsymbol{A}_1(t_0)=\boldsymbol{B}_1(t_0); \boldsymbol{A}_1'(t_0)=\boldsymbol{B}_1'(t_0); \boldsymbol{A}_1''(t_0)=\boldsymbol{B}_1''(t_0); \boldsymbol{A}_1'''(t_0)\neq \boldsymbol{B}_1'''(t_0). \tag{10}$$

Ruled surfaces are exactly in the second order of contact in their common generator $\boldsymbol{a}_{01}(t_0)= \boldsymbol{b}_{01}(t_0)$ (see Figure 1,b). From relations:

$$\boldsymbol{A}_1''=\left(H_A\cdot\boldsymbol{A}_2\right)'=-H_A^{\,2}\cdot\boldsymbol{A}_1+H_A'\cdot\boldsymbol{A}_2+H_A\cdot Q_A\cdot\boldsymbol{A}_3;$$
$$\boldsymbol{B}_1''=\left(H_B\cdot\boldsymbol{B}_2\right)'=-H_B^{\,2}\cdot\boldsymbol{B}_1+H_B'\cdot\boldsymbol{B}_2+H_B\cdot Q_B\cdot\boldsymbol{B}_3,$$

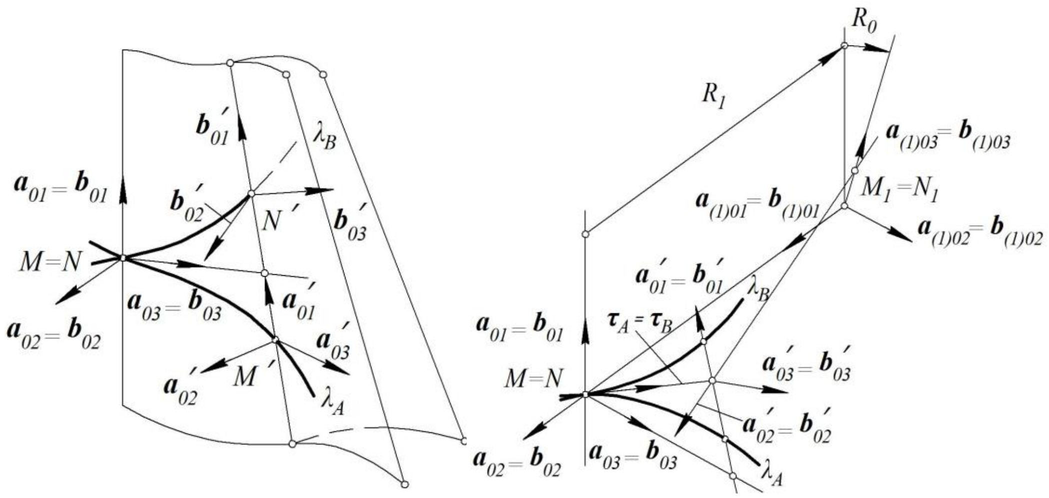
it follows that additionally to the result of contact of order $n = 1$ the following equalities are carried out:

$$H_A'=H_B'; Q_A=Q_B. \tag{11}$$

The element of the dual arc of ruled surface $\boldsymbol{A}_3(t)$ on the basis of derivation equation $\boldsymbol{A}'_3 = -Q_A\cdot\boldsymbol{A}_2$ may be expressed as $ds_{A3} = |\boldsymbol{A}'_3|\cdot dt = Q_A\cdot dt$. On the basis of the second equation of (11) it follows that $ds_{A3}=ds_{B3}$. Thus, we obtain the equality of elements of dual arcs of ruled surfaces $\boldsymbol{A}_3(t)$ and $\boldsymbol{B}_3(t)$. And, similarly, on the basis of equations (11) and $\boldsymbol{A}'_2 = -H_A\cdot\boldsymbol{A}_1 + Q_A\cdot\boldsymbol{A}_3$ it follows that $ds_{A2}=ds_{B2}$. We obtain the equality of elements of dual arcs of ruled surfaces $\boldsymbol{A}_2(t)$ and $\boldsymbol{B}_2(t)$.

If order n of contact of initial surfaces is equal 2 then the elements of dual arcs in pairs of surfaces $\boldsymbol{A}_i(t)$ and $\boldsymbol{B}_i(t)$, where i=1, 2, 3, are equal and calculated from common generators in these pairs.

The equality of elements of dual arcs is equivalent to the equality of distribution parameters in the common generators of these pairs of surfaces. In theory of ruled surfaces [3] it is known that $\dfrac{ds_{A2}}{ds_A}=\dfrac{1}{\rho_A}$, where ρ_A - a dual curvature radius of ruled surface $\boldsymbol{A}_1(t)$ at centre point M of its ruling. Since order $n = 2$ of contact is characterized by equality $ds_A=ds_B$ and $ds_{A2}=ds_{B2}$ it follows that at this contact ruled surfaces in their common generator $\boldsymbol{a}_{01}(t_0) = \boldsymbol{b}_{01}(t_0)$ have equal dual curvature radius $\rho_A=\rho_B$. Furthermore, since $\rho=\sin R$, where $R=R_0+\omega R_1$ is a dual angle between the generator of ruled surface $\boldsymbol{A}_1(t)$ and the generator of its first evolute (see Figure 1, b), it follows from equation $\rho_A=\rho_B$ that $R_A=R_B$ and the trihedrons of the first order evolutes coincide.



a b

Figure 1. Orders n = 1 (a) and n = 2 (b) of the contact.

2.4. Order n=3 of contact of ruled surfaces

$$\boldsymbol{A}_1(t_0)=\boldsymbol{B}_1(t_0);\boldsymbol{A}_1'(t_0)=\boldsymbol{B}_1'(t_0);\boldsymbol{A}_1''(t_0)=\boldsymbol{B}_1''(t_0);\ \boldsymbol{A}_1'''(t_0)=\boldsymbol{B}_1'''(t_0);\boldsymbol{A}_1^{\mathrm{IV}}(t_0)\neq\boldsymbol{B}_1^{\mathrm{IV}}(t_0). \tag{12}$$

Ruled surfaces are exactly in the third order of contact in their common generator $\boldsymbol{a}_{01}(t_0) = \boldsymbol{b}_{01}(t_0)$. From relations

$$\boldsymbol{A}_1'''= -3H_A\cdot H_A'\cdot\boldsymbol{A}_1+\left(H_A''-H_A{}^3-H_A\cdot Q_A{}^2\right)\cdot\boldsymbol{A}_2+\left(2H_A'\cdot Q_A+H_A\cdot Q_A'\right)\cdot\boldsymbol{A}_3;$$
$$\boldsymbol{B}_1'''= -3H_B\cdot H_B'\cdot\boldsymbol{B}_1+\left(H_B''-H_B{}^3-H_B\cdot Q_B{}^2\right)\cdot\boldsymbol{B}_2+\left(2H_B'\cdot Q_B+H_B\cdot Q_B'\right)\cdot\boldsymbol{B}_3,$$

it follows that additionally to the results of order n = 2 of contact the following equalities are carried out:

$$H_A''=H_B'';Q_A'=Q_B'. \tag{13}$$

Thus in order n=3 of contact the following equations hold:

$$\begin{gathered}\frac{ds_{A1}}{dt}=H_A=H_B=\frac{ds_{B1}}{dt};\frac{d^2s_{A1}}{dt^2}=H_A'=H_B'=\frac{d^2s_{B1}}{dt^2};\frac{d^3s_{A1}}{dt^3}=H_A''=H_B''=\frac{d^3s_{B1}}{dt^3};\\ \frac{ds_{A3}}{dt}=Q_A=Q_B=\frac{ds_{B3}}{dt};\frac{d^2s_{A3}}{dt^2}=Q_A'=Q_B'=\frac{d^2s_{B3}}{dt^2}.\end{gathered} \tag{14}$$

Central tangent vector $\boldsymbol{A}_3(t)$ has coordinates $\boldsymbol{A}_3(t)=\{x_3(t), y_3(t), z_3(t)\}$.

We define the derivatives of this vector based on the existence of functional relationship $s_{A1}(t)$:

$$\boldsymbol{A}_3'=\frac{d\boldsymbol{A}_3}{ds_{A1}}\cdot\frac{ds_{A1}}{dt}=\left\{\frac{dx_3}{ds_{A1}}\cdot\frac{ds_{A1}}{dt};\frac{dy_3}{ds_{A1}}\cdot\frac{ds_{A1}}{dt};\frac{dz_3}{ds_{A1}}\cdot\frac{ds_{A1}}{dt}\right\}.$$

In complex line geometry [3] the following equations are known:

$$\frac{dx_3}{ds_{A1}}=-x_2\cdot\mathrm{ctg}R;\frac{dy_3}{ds_{A1}}=-y_2\cdot\mathrm{ctg}R;\frac{dz_3}{ds_{A1}}=-z_2\cdot\mathrm{ctg}R, \tag{15}$$

where $\boldsymbol{A}_2(t)=\{x_2(t), y_2(t), z_2(t)\}$. We will consider the obtainment of a second derivative of vector $\boldsymbol{A}_3(t)$: $\boldsymbol{A}_3''=\left\{\frac{d^2x_3}{ds^2{}_{A1}}\cdot\frac{ds_{A1}}{dt}+\frac{dx_3}{ds_{A1}}\cdot\frac{d^2s_{A1}}{dt^2};\ \dots;\ \dots\right\}.$

Proceeding from equations (15) for vector $\boldsymbol{A}''_3$ we can write:

$$\boldsymbol{A}_3''(t)=\left\{\left(-\frac{x_{(1)2}}{\rho_{A1}}\operatorname{ctg}R+\frac{x_2}{\sin R}\cdot\frac{1}{r_{A1}}\right)\frac{ds_{A1}}{dt}+\left(-x_{(1)2}\operatorname{ctg}R\right)\frac{d^2s_{A1}}{dt^2};\ \ldots;\ \ldots\right\} \tag{16}$$

where $\boldsymbol{A}_{(1)2}=\{x_{(1)2}, y_{(1)2}, z_{(1)2}\}$, $\frac{1}{r_{A1}}=\frac{dR}{ds_{A1}}$, ρ_{A1} and r_{A1} – are dual radii of curvature and torsion of ruled surface $\boldsymbol{A}_1(t)$ in the generator of contacting. The analysis of the components of vector $\boldsymbol{A}''_3(t)$ in view of equations (14) and $dR+ds_{A(1)3}=0$, which is peculiar to the evolute of ruled surface $\boldsymbol{A}_1(t)$ [3], allows making certain conclusions. For order $n = 3$ of contact of the nondevelopable ruled surfaces the elements of dual arcs in pairs of surfaces $\boldsymbol{A}_{(1)i}(t)$ and $\boldsymbol{B}_{(1)i}(t)$, $i = 1, 2, 3$ are equal. Hence, the evolute of the first order has order n=2 of contact. In addition, the trihedrons of the evolute of the second order coincide and dual radii of curvature $r_{A1}=r_{B1}$ of $\boldsymbol{A}_1(t)$ and $\boldsymbol{B}_1(t)$ with parameter $t = t_0$ are equal.

3. Theoretical examples

As an example we will consider contact of ruled surface $\boldsymbol{A}_1\rightarrow\boldsymbol{R}_A(t,u)$ with elliptical hyperboloid $\boldsymbol{B}_1\rightarrow\boldsymbol{R}_B(t,u)$ (Figure 2):

$$\boldsymbol{A}_1\rightarrow\boldsymbol{R}_A(t,u)=\left\{A(\cos t+t\sin t)+\frac{u\cos t}{m};A(\sin t-t\cos t)+\frac{u\sin t}{m};1+Bt+\frac{u}{m}\right\},$$

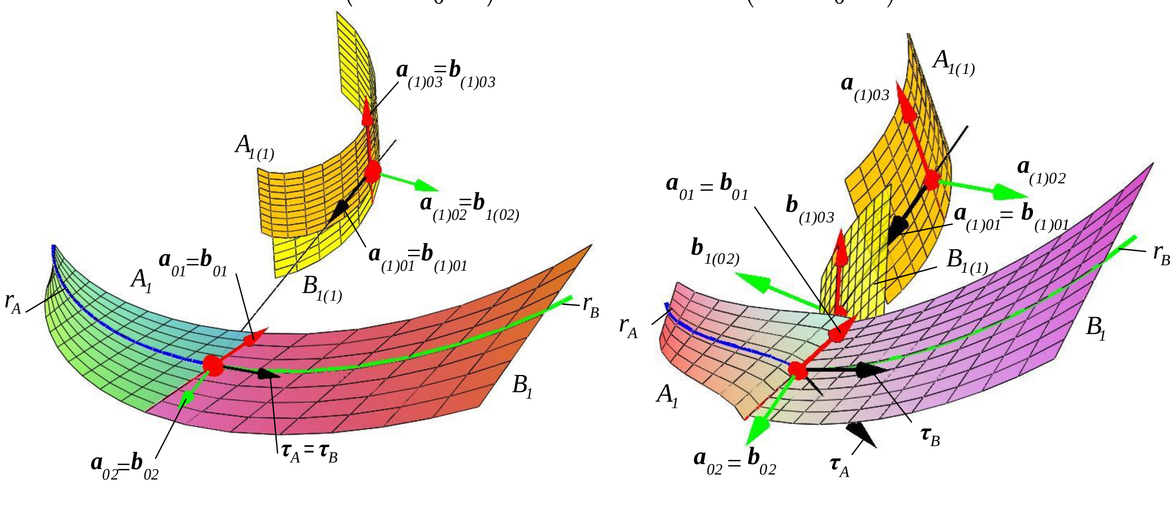
$$A=1; B=0.5; m=\sqrt{2}; 0\le t\le 2\pi; -\infty<u<\infty.$$

$$\boldsymbol{B}_1\rightarrow\boldsymbol{R}_B(t,u)=\left\{c_1\sin t+\frac{u\cos t}{\sqrt{1+c_3^{\,2}}};c_2\cos t+\frac{u\sin t}{\sqrt{1+c_3^{\,2}}};\frac{uc_3}{\sqrt{1+c_3^{\,2}}}\right\},$$

$$0\le t\le 2\pi, -\infty<u<\infty.$$

$$n=1:\qquad c_1=\frac{1-2t_0}{2c_3\left(3\cos^2 t_0-4\right)};c_2=\frac{-4(1-2t_0)}{2c_3\left(3\cos^2 t_0-4\right)};c_3=2;t_0=2.46.$$

$$n=2:\qquad c_1=\frac{2(1-t_0)-\cos^2 t_0\cdot(3-4t_0)}{2\left(2\cos^2 t_0-1\right)};c_2=-\frac{1-2t_0-\cos^2 t_0\cdot(3-4t_0)}{2\left(2\cos^2 t_0-1\right)};c_3=1;t_0=4.3.$$



a b

Figure 2. Theoretical examples of orders n = 1 (a) and n = 2 (b) of contact.

4. Practical results

The results of these researches were applied to designing of the working surfaces of the progressive tools for cultivation of the soil. In accordance with the mathematical model of tillage process 'soil-chisel plow' a ruled surface of the chisel plow's rack was designed, which increases the wear resistance and improves the quality of the soil loosening (Figure 3, chisel – 1, rack – 2, knife – 3, working surface – 4).

The developed mathematical theory allows designing the constituent ruled surfaces of technical products with pre-assigned geometric and technical conditions. The mathematical toolkit allows a flexible change of the shape of the surface depending on the type of soil to be treated.

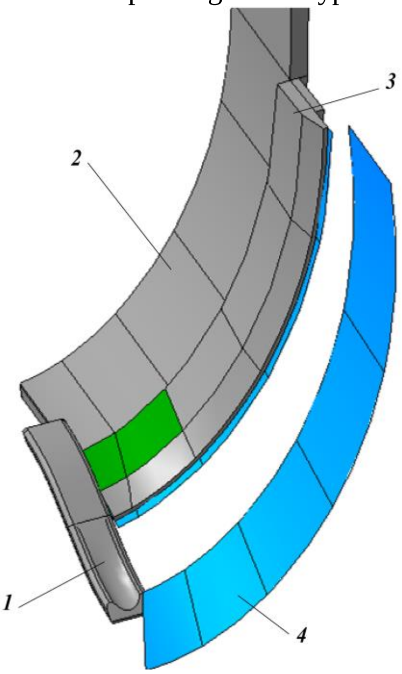


Figure 3. The working surface of the progressive tillage tool with order $n = 2$ of contact.

5. Conclusion

Order n of contact of nondevelopable ruled surfaces is considered. The notion of a dual vector of divergence on the common generator line of two contacting nondevelopable ruled surfaces is introduced via the dual vector calculus, which describes the manifolds of line geometry. To determine this vector we proved necessary and sufficient conditions for order n of contact on the common generator line. Thus, the primary orders of contact have been researched. The results of these researches can be applied in the design of the constituent ruled shells and in CAD systems via approximating issues.

References

[1] Pottmann H and Wallner J 2001 Computational Line Geometry (Springer-Verlag Berlin Heidelberg) p 563
[2] Blaschke W 1921 Vorlesungeon uber Differentialgeometrie. Elementare Differentialgeometrie (Springer Berlin 1. Auflage) vol **1** p 230
[3] Zeiliger D N Complex Line Geometry 1934 (Moscow-Leningrad) p 196
[4] Study E 1903 Geometrie der Dynamen (Teubner Leipzig) p 634
[5] Pottmann H, Peternell M and Ravani B 1999 *Computer-Aided Design* **31** 3–16