

Competing risk models in reliability systems, a gamma distribution model with bayesian analysis approach

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Abstract. In this paper our effort is to introduce the basic notions that constitute a competing risks models in reliability analysis using Bayesian analysis approach with Gamma distribution as our model and presenting their analytic methods. The Gamma distribution is widely used in reliability analysis and it is known as an natural extension of the exponential distribution. The cases are limited to the models with independent causes of failure, only the scale parameter is a random variable, and uniform prior distribution is used in our analysis. This model describes the likelihood function and follows with the description of the posterior function and the estimations of the point, interval, hazard function, and reliability. The net probability of failure if only one specific risk is present, crude probability of failure due to a specific risk in the presence of other causes, and partial crude probabilities are also included.

Keywords. Competing risks, Likelihood function, Posterior/prior function, Hazard function, and Net/Crude/partial crude probability

1. Introduction

The weakness of most reliability theories that the system and its components are described and explained as simply functioning or failed. However, depending on the age and the environment of the system (and/or its components), the failures may be from various causes. Suppose that units under observation experience any one several distinct failure types. Time to failure and cause of failure are recorded. Failure may be correspond to breakdown of a component where there are some possible causes of failure such as tread wear, puncture, or defective side walls of an automobile tire. In engineering area this phenomena is called as *competing risks*. Woman who is using birth control device might facing several risks such as getting pregnant accidentally, removal the device for medical or personal reasons, and the device rejection. In this case we might interested to estimate the length of time that woman to use this device when the rejection of it can be removed from the system. In bio-medicine and engineering areas, to describe this phenomena, people use the term *competing risks*. The competing risks theory describes how the performance of a system affected by many causes of failure that act together. Applications of this theory also occur in actuarial and demography sciences and it is called as *multiple-decrement analysis*.

Estimation of the parameters of the assumed failure models based on the classical sampling theory has been used broadly in many reliability areas. However, this classical methods have been observed



unsatisfactory for many instances especially when analysis based on the scarcity of data or small sample size. The increasing cost of reliability test and the changing of engineering design also contribute to this problem. In contrast, Bayesian estimation methods ability to combine past knowledge or experience in the form of an apriori distribution with life test data to accomplish the task with smaller sample sizes allow us to cope with that problem.

Competing risks models and Bayesian estimation have been studied since the middle of 17th century by numerous scientists and researchers. These models have been used extensively in area of bio-medicine but only a few in engineering; the estimation methods, however, have been classical. Bayesian estimation has been used widely in reliability analysis but the application so far is limited to single failure-mode models. The combination of the two is the contribution of this paper.

In this paper we select our investigation by using the Gamma distribution as our model. We assume here all causes of failure are independent. This distribution is one of some other common failure models such as Weibull, exponential, Normal, Log-normal, and Inverse Gaussian models. This distribution has been widely used to measure the reliability of electrical, mechanical, and electro-mechanical systems. This distribution is also more appropriate to model the degradation of materials (especially in the case of fatigue data) [1]. It is also known as an extension of the exponential distribution and often used as a prior distribution. This paper begins with the description of the likelihood function of Gamma distribution model and follows by the description of its posterior function. A uniform prior is used in our analysis for each model and it is followed by the estimation of the point, interval, hazard function, and reliability estimations. The net, crude, and partial crude probabilities are also included.

2. Literature Review

In 1760 Daniel Bernoulli tried to determine mathematically the effect of elimination of smallpox to a population mortality structure at different ages. This was considered as one of the first use of the theory of competing risks in bio-medicine area. This theory has been a great of interest and importance since the 19th century in which most of currently techniques were developed.

Numerous scientists [2,3,4,5,6,7] have studied this theory. They used the terminology of competing risks to describe, analyze and criticize and review some of these models. This technique was named as the theory of competing risks [3] for the first time after the work of Mahekan [2] who was the first scientist to formulate and studying its practical applications. The standard competing risks have been presented and the individual mortality risks that “compete” in an individual life have been studied [5]. There are three types of probability of failure from a specific cause have been described and classified i.e. net, crude and partial crude probabilities [5]. Some other scientists, who also contributed to this theory, studied the process of illness and death in which individuals exposed to risks of death that compete to each other continuously [8,9,10]. The failure-modes and mixed-population approaches related to reliability was described briefly [8]. There was a study on historical review and literature of actuarial method to competing risks and that presented the statistical contributions to this theory [10]. And there was also a discussion on the methods used in analyzing survival data with competing risks by using the SAS software to implement the appropriate nonparametric methods for estimating cumulative incidence functions and for testing differences between these functions in multiple groups [11]. For the risk-based design of complex engineering systems, a study of the challenges in the design process, the mathematical modeling of their topological architecture, and their unique behaviors was proposed [12]. A model for the accelerated degradation data from plastic substrate active matrix light-emitting diodes (AMOLEDs), along with sensitivity analysis, was proposed to estimate the reliability of the model with competing risks analyses from linear degradation and failure time data [13].

The basis for the famous Bayesian statistical inference was provided and published in 1763 by Rev. Thomas Bayes (1702-1761). Since that time, numerous scientists have contributed and provided the philosophical basis for this method and its use in many areas of application and research. The importance of this method was recognized and the Bayesian inference alone seems to offer the possibility of sufficient flexibility to allow reaction to scientific complexity free from impediment from purely technical limitation [14]. A new class of prior distributions for reliability growth tests under

Binomial data under the monotone model was introduced together with other available priors [15]. Comparisons are made by using two examples. By taking the advantage of graphic representation and uncertainty reasoning of Bayesian network, a method of reliability modeling and assessment of multi-state system with common cause failure was proposed [16]. It took into account of the influence of common cause failure to system reliability and the widespread presence of multi-state system in engineering practices. A Bayesian framework was developed to assess the reliability and performance of multi-state systems that consists of multiple multi-state components of which the degradation follows a Markov process [17]. For analyzing lifetime data obtained from accelerated life testing experiments model under Weibull lifetimes, a Bayesian analysis of a simple step-stress was used [18]. Monte Carlo simulations were performed to see the effectiveness of the proposed method, and a data set has been analyzed for illustrative purposes.

Using Gamma distribution as a model, a study has been done that dealt with the Bayesian estimation for reliability function and scale parameter in which shape parameter is known [19]. This model was also described for both parameters are random variable [20]. Under squared and absolute error loss functions, Bayesian point estimators of scale parameter and reliability function for Gamma distribution model was derived [21]. Along with some other models/methods, a method based on performance Gamma degradation data to assess reliability of high reliable long life time products based on Bayesian updating was proposed [22]. Other models/methods that have been introduced in this area are such as the use of Bayesian analysis of Gamma model with Laplace approximation to provide a solution for non availability of simple and closed forms of reliability and hazard functions of gamma model [23] and the use of Bayes estimators of the parameter of the exponentiated gamma distribution under general entropy loss function, squared error loss function and also derived its maximum likelihood estimator for progressive type II censored data with Binomial removals [24].

3. Gamma Distribution and Its Likelihood Function

The Gamma distribution is widely used in reliability analysis and it is known as a natural extension of the exponential distribution. This distribution is also often used as a prior distribution.

Let us consider to the case of a random sample from an uncensored life test. For a single mode case, if t_l are the failure times from a random sample of n items according to a Gamma distribution with the shape parameter κ and scale parameter θ , then the statistic $\sum_{l=1}^n t_l$ also follows a Gamma distribution with the shape parameter $n\kappa$ and scale parameter θ , where $l = 1, \dots, n$, and $\kappa > 0$. We can extend this process to the competing risks model.

The individual p.d.f. of the length of life t_i due to cause C_i is [25]

$$g_i(t_i; n_i\kappa_i, \theta_i) = \frac{1}{\theta_i^{n_i\kappa_i} \Gamma(n_i\kappa_i)} t_i^{n_i\kappa_i-1} \exp(-t_i/\theta_i), \quad (1)$$

where $t_i, n_i\kappa_i, \theta_i > 0$. The shape parameter and scale parameters are $n_i\kappa_i$ and θ_i , respectively.

In this paper, we use the uniform prior distribution in our analysis. Also we limit our model to the case where only the scale θ_i parameter is a random variable. The case with both scale and shape parameters are random variable is also possible to be developed.

Let us consider the case in which all lengths of life and associated causes of failure are known. The individual likelihood function is given by [25]

$$L_i(\theta_i) \propto \frac{1}{\theta_i^{n_i\kappa_i} \Gamma(n_i\kappa_i)} \left(\prod_{j=1}^{n_i} t_{ij}^{n_i\kappa_i-1} \right) \exp(-T/\theta_i), \quad (2)$$

where the statistic

$$T = \sum_{i=1}^k \sum_{j=1}^{n_i} t_{ij}, \quad (3)$$

is sufficient for estimation of the scale parameter θ_i .

4. The Posterior and Uniform Prior Functions

The individual posterior distribution is given by

$$g_i(\theta_i | t_{ij}) = \frac{(1/\theta_i)^{n_i \kappa_i} e^{-(T/\theta_i)} f_i(\theta_i)}{\int_0^\infty (1/\theta_i)^{n_i \kappa_i} e^{-(T/\theta_i)} f_i(\theta_i) d\theta_i}, \quad (4)$$

where $f_i(\theta_i)$ is the individual prior distribution.

The individual uniform prior distribution is as follows

$$f(\theta_i) = \begin{cases} \frac{1}{b_i - a_i} & a_i < \theta_i < b_i, \\ 0 & \text{elsewhere} \end{cases} \quad (5)$$

The individual posterior distribution in equation (4) becomes

$$g_i(\theta_i | t_{ij}) = \frac{(1/\theta_i)^{n_i \kappa_i} \exp(-T/\theta_i)}{\Gamma(n_i \kappa_i - 1, T/a_i) - \Gamma(n_i \kappa_i - 1, T/b_i)}, \quad a_i < \theta_i < b_i, \quad (6)$$

where $\Gamma(\cdot)$ represents the standard incomplete Gamma function.

5. Point Estimation

The individual posterior mean is computed to be

$$E_i(\theta_i | t_{ij}) = \frac{T[\Gamma(n_i \kappa_i - 2, T/a_i) - \Gamma(n_i \kappa_i - 2, T/b_i)]}{\Gamma(n_i \kappa_i - 1, T/a_i) - \Gamma(n_i \kappa_i - 1, T/b_i)}. \quad (7)$$

with the individual posterior variance

$$\text{Var}(\theta_i | t_{ij}) = \left\{ \frac{T[(n_i \kappa_i - 3, T/a_i) - \Gamma(n_i \kappa_i - 3, T/b_i)]}{\Gamma(n_i \kappa_i - 1, T/a_i) - \Gamma(n_i \kappa_i - 1, T/b_i)} - \left(\frac{T[\Gamma(n_i \kappa_i - 2, T/a_i) - \Gamma(n_i \kappa_i - 2, T/b_i)]}{\Gamma(n_i \kappa_i - 1, T/a_i) - \Gamma(n_i \kappa_i - 1, T/b_i)} \right)^2 \right\}. \quad (8)$$

The individual hazard function is given by

$$h_i(t) = \frac{t^{n_i \kappa_i - 1} \exp\{-t/E_i(\theta_i | t_{ij})\}}{[E_i(\theta_i | t_{ij})]^{n_i \kappa_i} [\Gamma(n_i \kappa_i) - \Gamma(n_i \kappa_i, t/E_i(\theta_i | t_{ij}))]}. \quad (9)$$

For $n_i \kappa_i$ integer, this function has a closed form as follows

$$h_i(t) = \frac{t^{n_i \kappa_i - 1}}{[E_i(\theta_i | t_{ij})]^{n_i \kappa_i} \Gamma(n_i \kappa_i) \sum_{m=0}^{n_i \kappa_i - 1} (t/E_i(\theta_i | t_{ij}))^m / m!}. \quad (10)$$

The hazard function is increasing for $n_i \kappa_i > 1$, a constant for $n_i \kappa_i = 1$, and decreasing for $n_i \kappa_i < 1$. For a constant $n_i \kappa_i$, the Gamma distribution reduces to the exponential distribution.

The hazard function for the system is as follows

$$h(t) = \sum_{i=1}^k \frac{t^{n_i \kappa_i - 1} \exp\{-t/E_i(\theta_i | t_{ij})\}}{[E_i(\theta_i | t_{ij})]^{n_i \kappa_i} [\Gamma(n_i \kappa_i) - \Gamma(n_i \kappa_i, t/E_i(\theta_i | t_{ij}))]}. \quad (11)$$

For $n_i \kappa_i$ integer, this function becomes

$$h(t) = \sum_{i=1}^k \frac{t^{n_i \kappa_i - 1}}{[E_i(\theta_i | t_{ij})]^{n_i \kappa_i} \Gamma(n_i \kappa_i) \sum_{m=0}^{n_i \kappa_i - 1} (t/E_i(\theta_i | t_{ij}))^m / m!}. \quad (12)$$

6. Interval Estimation

The $100(1-\gamma)\%$ *two-sided Bayes confident interval (TBCI)* equations for each individual posterior mean, for lower bound $\theta_{i,L}$ and upper bound $\theta_{i,U}$, are given by

$$\text{Prob}(\theta_i \leq \theta_{i,L} | t_{ij}) = \frac{\Gamma(n_i \kappa_i - 1, T/a_i) - \Gamma(n_i \kappa_i - 1, T/\theta_{i,L})}{\Gamma(n_i \kappa_i - 1, T/a_i) - \Gamma(n_i \kappa_i - 1, T/b_i)} = \frac{\gamma_i}{2}, \quad (13)$$

and

$$\text{Prob}(\theta_i \geq \theta_{i,U} | t_{ij}) = \frac{\Gamma(n_i \kappa_i - 1, T/\theta_{i,U}) - \Gamma(n_i \kappa_i - 1, T/b_i)}{\Gamma(n_i \kappa_i - 1, T/a_i) - \Gamma(n_i \kappa_i - 1, T/b_i)} = \frac{\gamma_i}{2}. \quad (14)$$

To find the TBCI for each θ_i , these equations can be solved numerically.

7. Reliability Estimation

The reliability function for the system is given by

$$R_{sys}(t | \theta_i' s) = \prod_{i=1}^k \frac{\Gamma(n_i \kappa_i) - \Gamma\{n_i \kappa_i / E_i(\theta_i | t_{ij})\}}{\Gamma(n_i \kappa_i)}. \quad (15)$$

For $n_i \kappa_i$ integer, equation (15) becomes

$$R_{sys}(t | \theta_i' s) = \prod_{i=1}^k \left\{ \sum_{m=0}^{n_i \kappa_i - 1} \frac{t/E_i(\theta_i | t_{ij})^m \exp[-t/E_i(\theta_i | t_{ij})]}{m!} \right\} \quad (16)$$

8. The Net, Crude, and Partial Crude Probabilities

This section describes the estimation of the Chiang's probabilities of failure in the Gamma model [5].

Suppose that C_i is the only risk present. The value of the *net probability* q_{net} can be obtained as follows

$$q_{i,net} = 1 - \exp\left\{-\int_0^t h(x) dx\right\} = 1 - \exp\left\{-\int_0^t \sum_{i=1}^k \frac{x^{n_i \kappa_i - 1} \exp\{-x/E_i(\theta_i | t_{ij})\}}{[E_i(\theta_i | t_{ij})]^{n_i \kappa_i} [\Gamma(n_i \kappa_i) - \Gamma(n_i \kappa_i, x/E_i(\theta_i | t_{ij}))]} dx\right\} \quad (17)$$

The *crude probability* of failure can be obtained as follows

$$Q_{i,crude} = \int_0^t \left\{ h_i(x) \exp\left(-\int_0^x h(y) dy\right) \right\} dx. \quad (18)$$

The hazard functions $h_i(t)$ and $h(t)$ are given by equations (9) and (11), respectively.

The *partial crude probability* of failure if risk C_1 is eliminated from the system while $k-1$ risks remain is given by

$$Q_{i,1;prt,crude} = \int_0^t h_i(t) \exp\left\{-\int_0^x [h(t) - h_1(t)] dy\right\} dx. \quad (19)$$

with hazard functions $h_i(t)$ and $h(t)$ are given by equations (9) and (10), respectively. Equation (19) can be modified if we want to eliminate l risks for $1 < l < k$ from the system.

9. Conclusions

The primary objective is to investigate the applications of the Bayesian estimation methods to competing risks of the independent risks for a selected Gamma distribution as our model. We introduce the basic notions that constitute competing risks models in reliability analysis using Bayesian approach, and presenting their analytic methods.

There are some models that are open to be developed in this area, depending on the problems of interest. In this paper we select our investigation by using the Gamma distribution. This distribution is widely used in reliability analysis and is one of some other common failure models such as Weibull,

exponential, Normal, Log-normal, and Inverse Gaussian models. It is also known as an extension of the exponential distribution and often used as a prior distribution. It should be noted that the choice of the time to failure distribution functions depend on the choice of its reliability measures that is meaningful and useful to the problem of interest. Different pattern of operating life of a device may required to have a specific probability of successfully performing its required function.

We limit our model to the case where only the scale parameter is a random variable. The case with both scale and shape parameters are random variable is also possible to be developed. This model describes the likelihood function and follows with the description of the posterior function. Although other possible priors can be used, in this paper we select an uniform prior for our analysis for each model. The choice of this prior, as well as others, generally satisfies criterias as follows: able to be tracked analytically, flexible and rich, and can be interpreted easily [26]. The estimations are provided for individual posterior function (the individual mean time to failure / MTTF) and its variance and followed by the estimations of the hazard and the reliability functions. Based on these estimations, others such as the estimations for reliable or design life for a specific reliability and the probability of survival between a specific interval of time can also be calculated. One important part of competing risks analysis is its ability to elaborate the net probability if only one risk is present and the crude / partial crude probabilities if one or more risk is eliminated from the system while other risks remain. The author realizes that for all descriptions of this model above need a real or simulated data to test and prove the validity of the model. But the limitation of time to obtain or to conduct a simulation for such a data has put this matter for further investigation without eliminating the importance of this model. Another important measures of this analysis is to include the dependent risks in our model for further research.

10. References

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