

On the theory of plasticity, associated with a new integral characteristic of shearing stresses

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Abstract. A new invariant of stress tensor is introduced – the mean shearing stress, resulted from the integration with respect to Mohr's circle. The invariant is used to lay down the terms of plasticity. Determining equations are written on the basis of associated flow law. Rigid variants of the model and elastoplastic ones are obtained. Characteristic surfaces with normals, coinciding with main stresses direction are demonstrated for the rigid-plastic variant.

Introduction

Plastic deformation of solids has a shearing character and is associated with shearing stresses, existing in the solid. This fact is confirmed by the data of numerous experiments. It is reflected in corresponding theoretical constructs. For instance, the equation of plane deformation of perfectly plastic bodies is of a hyperbolic type. Characteristics of stress field are similar to those of field of velocities – glide lines. For the latter ones Geiringer's ratios are characteristic [1]. These properties of the model confirm the shearing nature of plasticity and are arguments in support of adequacy of classical theory [2-4].

The situation is more complicated in three-dimensional space. In general, the conditions of plasticity are to consider both maximum shearing stress (Tresca criterion), and intermediate stresses, effecting in other places of a volumetric unit. In this respect numerous experimental [5-9] and theoretical [10] investigations have been conducted for various construction materials and for rocks. Nevertheless, the problem has not been solved completely yet. Efforts are being made both experimentally and theoretically [11, 12]. This paper provides the consideration of plasticity theory variant, which is based on a new integral characteristic of shearing stresses, effecting in various places of a volumetric unit of a body to be deformed. A classical integral characteristic of shearing stresses is associated with the second invariant of stress tensor. It appears rather naturally when investigating proper values of stress tensor. The well-known paper [13] has demonstrated that this measure of shearing stresses can be made as the root mean square calculated according to the surface of infinitely small elementary sphere. It is the sphere, which has all symmetrical and equivalent directions of places. One more method of averaging can be noted, and involves all quite equivalent values of shearing stresses τ_n . It is averaging in the plane of Mohr's circle (σ_n, τ_n) . Here σ_n, τ_n are normal and shearing stresses on the place with the normal \bar{n} . We know, that every possible \bar{n} is in line with the area restricted by three Mohr's semicircumferences. Averaging in regard with mentioned semicircumferences has the following result [14, 15]:



$$\langle \tau_n \rangle = \frac{1}{2\pi} \frac{(\sigma_1 - \sigma_3)^2 + (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2}{\sigma_1 - \sigma_3}. \quad (1)$$

Here $\sigma_3 < \sigma_2 < \sigma_1$ are ranged main stresses. Let us consider the model of plastic deformation, associated with the loading surface or yield surface of the form:

$$f(\sigma_1, \sigma_2, \sigma_3) = \langle \tau_n \rangle - k = 0, \quad (2)$$

where k is either a set constant, or a pre-set hardening function. As it is easy to see, if a point $(\sigma_1^0, \sigma_2^0, \sigma_3^0)$ is an element of the surface (2), so the point $(\sigma_1^0 + C, \sigma_2^0 + C, \sigma_3^0 + C)$ is an element of this surface too, where $C = \text{const}$. Therefore, the surface (2) is a cylinder with a straight parallel line $\sigma_1 = \sigma_2 = \sigma_3$. The section of the cylinder by the orthogonal plane is depicted in Fig. 1. In comparison with

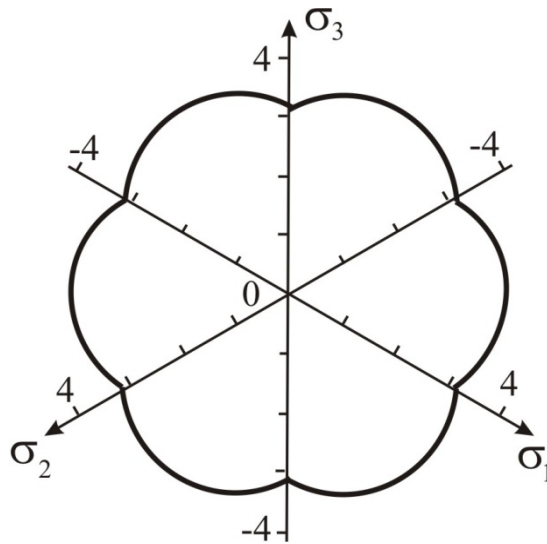


Figure. 1. Loading surface section by deviatoric plane

Tresca prism and Mises cylinder the surface (2) is nonconvex one. Therefore, Mises principle of the maximum is not relevant for it. [11]. However, it can't interfere with development of the theory associated with loading surface (2). The principle of maximum isn't as obligatory as conservation law. Furthermore, it isn't observed by materials with internal friction meeting the unassociated flow law. Moreover, this principle can be reformulated to make solely associated flow law follow from it but not convexity of a loading surface. Let us admit that the rate of energy dissipation in the course of plastic deformation is maximal for actual stress state among all stress states, which are close to the actual one and are in line with the given condition of plasticity. The main point of it is, that we don't consider all states meeting the plasticity condition, but those ones, which are close to the actual stress state. In this context we can say that the introduced principle is a local one as if compared to Mises principle.

Plastic model

Let us consider the equation of deformation. Let a particular deflected mode be obtained at a definite point of time t . This mode can be thought as the known one. Let us set up an equation of stress, deformation and shearing increment. Let us suppose that the body is a work-hardening one. It means that parameter k can change in the condition (2) in the course of deformation. The conditions of dynamic loading and un-loading are as follows:

$$\Delta f(\sigma_1, \sigma_2, \sigma_3) > 0 - \text{loading}, \quad \Delta f(\sigma_1, \sigma_2, \sigma_3) < 0 - \text{un-loading}. \quad (3)$$

As usual $\Delta f = 0$ coincides with neutral loading. When un-loading, there is no change in plastic deformation. When dynamic loading we have an associated flow law according to (3) and the introduced above reformulated principle of maximum. In terms of this law for work-hardening body we have

$$\begin{aligned}\Delta \varepsilon_1^p &= \frac{\partial f}{\partial \sigma_1} \Delta \lambda = \left[\frac{1}{\pi} \frac{2\sigma_1 - \sigma_2 - \sigma_3}{\sigma_1 - \sigma_3} - \frac{k}{\sigma_1 - \sigma_3} \right] \Delta \lambda, \\ \Delta \varepsilon_2^p &= \frac{\partial f}{\partial \sigma_2} \Delta \lambda = \frac{2\sigma_2 - \sigma_1 - \sigma_3}{\sigma_1 - \sigma_3} \Delta \lambda, \\ \Delta \varepsilon_3^p &= \frac{\partial f}{\partial \sigma_3} \Delta \lambda = \left[\frac{1}{\pi} \frac{2\sigma_3 - \sigma_1 - \sigma_2}{\sigma_1 - \sigma_3} + \frac{k}{\sigma_1 - \sigma_3} \right] \Delta \lambda,\end{aligned}\tag{4}$$

where $\Delta \lambda$ is a parameter, its equations are given consideration to later, $\Delta \varepsilon_i^p$ – increments of main plastic deformations, $i = 1, 2, 3$.

As it can be seen from (4), the material is practically incompressible:

$$\Delta \varepsilon_1^p + \Delta \varepsilon_2^p + \Delta \varepsilon_3^p = 0.$$

In terms of associated flow law plastic incompressibility is caused by the absence of internal friction in material, i.d. plasticity condition doesn't depend on the first invariant of stress tensor. A special case, when deformation comes practically to the plane one, is of high relevance for applications and the theory.

Let us admit for the case of plane deformation $\Delta \varepsilon_2^p = 0$. Then, according to (4) we have $\sigma_2 = (\sigma_1 + \sigma_3) / 2$. Plasticity condition (2) transforms into Tresca condition:

$$\langle \tau_n \rangle = \frac{3}{2\pi} \frac{\sigma_1 - \sigma_3}{2} = k.$$

Therefore, the theory under consideration in a particular case transforms into classical one [1-4]. Thus, the consistency requirement, which is usually specified for a new theory is met in this case. Let us address the issue of similarity of stress tensor to the plastic deformation rate. The definition states that tensor similarity is determined by comparison of Lode-Nadai coefficients for various tensors. It can be seen from (4):

$$\mu_\varepsilon = \frac{2\Delta \varepsilon_2^p - (\Delta \varepsilon_1^p + \Delta \varepsilon_3^p)}{\Delta \varepsilon_1^p - \Delta \varepsilon_3^p} = \frac{2\sigma_2 - \sigma_1 - \sigma_3}{\sigma_1 - \sigma_3 - \frac{2}{3}\pi k}.$$

Therefore, in conditions $k \neq 0$, $\mu_\varepsilon \neq \mu_\sigma$, i.d. there is no similarity of tensors. The degree of deviation from similarity is determined by the obtained limit of plasticity k . The combination of tensor components relevant for Lode-Nadai coefficient is important for the theory of plasticity. It is widely applied when processing experimental data and developing theories; and, to be more precise, as or instead of the third invariant of stress tensor. However, no papers (except [14, 15]) are at authors' disposal, where this combination grounds on reasonably general theoretical pre-conditions. In [14, 15] and equalities (4) this combination is the result of admitted method to average shearing stresses. Moreover, we have above accepted $\langle \tau_n \rangle$ as an integral characteristic of shearing stresses. Hence, the condition of hardening can be naturally written in terms $\langle \tau_n \rangle - \Delta \lambda$:

$$\Delta \langle \tau_n \rangle = G \Delta \lambda.$$

Here G is a local module of plastic shear. Explicitly, the condition takes on form:

$$\begin{aligned} & \left[\frac{2\sigma_1 - \sigma_2 - \sigma_3}{\pi(\sigma_1 - \sigma_3)} - \frac{k}{\sigma_1 - \sigma_3} \right] \Delta\sigma_1 + \frac{2\sigma_2 - \sigma_1 - \sigma_3}{\pi(\sigma_1 - \sigma_3)} \Delta\sigma_2 + \\ & + \left[\frac{2\sigma_3 - \sigma_1 - \sigma_2}{\pi(\sigma_1 - \sigma_3)} + \frac{k}{\sigma_1 - \sigma_3} \right] \Delta\sigma_3 = G\Delta\lambda. \end{aligned} \quad (5)$$

In terms of associated flow law we suppose that tensor of plastic deformation increment is coaxial to stress tensor. Let us make use of [11] and formulate a condition of two tensors σ_{ij} and $\Delta\epsilon_{ij}^p$ coaxiality as follows:

$$\begin{aligned} & (\sigma_{11}\Delta\epsilon_{12}^p - \sigma_{12}\Delta\epsilon_{11}^p) + (\sigma_{12}\Delta\epsilon_{22}^p - \sigma_{22}\Delta\epsilon_{12}^p) + (\sigma_{13}\Delta\epsilon_{23}^p - \sigma_{23}\Delta\epsilon_{13}^p) = 0, \\ & (\sigma_{12}\Delta\epsilon_{13}^p - \sigma_{13}\Delta\epsilon_{12}^p) + (\sigma_{22}\Delta\epsilon_{23}^p - \sigma_{23}\Delta\epsilon_{22}^p) + (\sigma_{23}\Delta\epsilon_{33}^p - \sigma_{33}\Delta\epsilon_{23}^p) = 0, \\ & (\sigma_{13}\Delta\epsilon_{11}^p - \sigma_{11}\Delta\epsilon_{13}^p) + (\sigma_{23}\Delta\epsilon_{12}^p - \sigma_{12}\Delta\epsilon_{23}^p) + (\sigma_{33}\Delta\epsilon_{13}^p - \sigma_{13}\Delta\epsilon_{33}^p) = 0. \end{aligned} \quad (6)$$

Now let us address to the formula of mean shearing stress (1). The structure of this formula predetermines the associated flow law and all further constructs. In the numerator (1) there is the square of stress intensity. It is rather easy to be calculated through stress components in any axes. Unfortunately, the denominator can't be written as visually graspable formulae through stresses in random coordinates. That is why, the increment of main stresses are to be considered as unknown functions. We have to apply the following equation in order to state the relation of main stresses increment to the increment in random axes:

$$\sigma^3 - J_1\sigma^2 + J_2\sigma - J_3 = 0, \quad (7)$$

where

$$\begin{aligned} J_1 &= \sigma_{11} + \sigma_{22} + \sigma_{33}, \\ J_2 &= \sigma_{11}\sigma_{33} + \sigma_{22}\sigma_{33} + \sigma_{11}\sigma_{22} - \sigma_{13}^2 - \sigma_{12}^2 - \sigma_{23}^2, \\ J_3 &= \sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{23}\sigma_{13}\sigma_{12} - \sigma_{11}\sigma_{23}^2 - \sigma_{22}\sigma_{13}^2 - \sigma_{33}\sigma_{12}^2. \end{aligned} \quad (8)$$

first, second and third stress invariants.

As far as we know, this equation has three real roots, which are the main stresses. Root values can be expressed principally through stresses (8) according to Cardano formula. Then we deal with increments and use the result in the restricted model of plasticity. The needed computations are quite cumbersome. It is better to address to increments in equation (7) and then use equations (8). So, we have:

$$3\sigma^2\Delta\sigma - 2J_1\sigma\Delta\sigma - \sigma^2\Delta J_1 + J_2\Delta\sigma + \sigma\Delta J_2 - \Delta J_3 = 0$$

Hence,

$$\Delta\sigma = A(\sigma)\Delta J_1 + B(\sigma)\Delta J_2 + C(\sigma)\Delta J_3,$$

where

$$\begin{aligned} A(\sigma) &= \frac{\sigma_1}{3\sigma^2 - 2\sigma J_1 + J_2}, \\ B(\sigma) &= -\frac{\sigma}{3\sigma^2 - 2\sigma J_1 + J_2}, \\ C(\sigma) &= \frac{1}{3\sigma^2 - 2\sigma J_1 + J_2}. \end{aligned}$$

Addressing to increments in (8) we obtain

$$\Delta\sigma_i = D_{11}^i\Delta\sigma_{11} + D_{22}^i\Delta\sigma_{22} + D_{33}^i\Delta\sigma_{33} + D_{12}^i\Delta\sigma_{12} + D_{13}^i\Delta\sigma_{13} + D_{23}^i\Delta\sigma_{23}, \quad (9)$$

where

$$\begin{aligned}
D_{11}^i &= A^i + B^i(\sigma_{22} + \sigma_{33}) + C^i(\sigma_{22}\sigma_{33} - \sigma_{23}^2), \\
D_{22}^i &= A^i + B^i(\sigma_{11} + \sigma_{33}) + C^i(\sigma_{11}\sigma_{33} - \sigma_{13}^2), \\
D_{33}^i &= A^i + B^i(\sigma_{11} + \sigma_{22}) + C^i(\sigma_{11}\sigma_{22} - \sigma_{12}^2), \\
D_{12}^i &= -2B^i\sigma_{12} + 2C^i(\sigma_{23}\sigma_{13} - \sigma_{33}\sigma_{12}), \\
D_{13}^i &= -2B^i\sigma_{13} + 2C^i(\sigma_{23}\sigma_{12} - \sigma_{22}\sigma_{13}), \\
D_{23}^i &= -2B^i\sigma_{23} + 2C^i(\sigma_{13}\sigma_{12} - \sigma_{11}\sigma_{23}), \\
A^i &= A(\sigma_i), \quad B^i = B(\sigma_i), \quad C^i = C(\sigma_i), \quad i = 1, 2, 3.
\end{aligned}$$

When stating the relations (9) we took into account that stresses σ_{ij} formed a symmetric tensor. Therefore, the same relations (9) are relevant for plastic deformations, where σ is to be replaced by ε^p with corresponding indexes:

$$\Delta \varepsilon_i^p = D_{11}^i(\varepsilon_i^p)\Delta \varepsilon_{11}^p + D_{22}^i\Delta \varepsilon_{22}^p + D_{33}^i\Delta \varepsilon_{33}^p + D_{12}^i\Delta \varepsilon_{12}^p + D_{13}^i\Delta \varepsilon_{13}^p + D_{23}^i\Delta \varepsilon_{23}^p \quad (10)$$

Three equations of equilibrium are to be added to the obtained equations

$$\frac{\partial \Delta \sigma_{ij}}{\partial x_i} + \Delta X_i = 0, \quad (11)$$

where ΔX_i – components of mass force increment. If we exclude elastic deformations, the equations can be closed by relations of deformation increment to displacement vector components:

$$\Delta \varepsilon_{ij}^p = \frac{1}{2} \left(\frac{\partial \Delta u_i}{\partial x_j} + \frac{\partial \Delta u_j}{\partial x_i} \right). \quad (12)$$

Thus, a restricted model is formulated and includes 22 equations in the following 22 unknowns: $\Delta \sigma_{ij}, \Delta \varepsilon_{ij}^p, \Delta \sigma_i, \Delta \varepsilon_i^p, \Delta u_i, \Delta \lambda$. This model is a rigidly plastic one because elastic deformations aren't taken into consideration.

Type of equations

The fact that the system of equations is kinematically determined in a rigidly plastic variant is of some interest. In fact, we substitute relations (12) into coaxiality equation (6). As the result we obtain three equations in variables $\Delta u_1, \Delta u_2, \Delta u_3$. We know all coefficients in this system from the previous stage. The system is invariant with respect to coordinate rotation. Therefore, we can assume that characteristic surface touches the plane $x_3 = 0$ in the origin of coordinates. Let us know Cauchy data on the surface. Consequently, we know in-situ derivatives with respect to x_1 and x_2 . Therefore [3], the system of differential equations can be treated as the algebraic system with respect to derivatives $\Delta u_1, \Delta u_2, \Delta u_3$ of normal $0x_3$. So, we need to equate to zero the determinant of three derivatives $\frac{\partial \Delta u_i}{\partial x_3}$ to calculate a characteristic surface. As the result we obtain

$$\sigma_{13}\sigma_{23}(\sigma_{22} - \sigma_{11}) + \sigma_{12}(\sigma_{13}^2 - \sigma_{23}^2) = 0.$$

The equation has one solution of the kind $\sigma_{13} = 0, \sigma_{23} = 0$. Thus, the normal to the surface coincides with the direction of main stress.

Consideration of elastic deformations

In some cases, when plastic flows are in the first stages of their development, elastic deformations can be easily compared with plastic deformations. Therefore, they are necessary to pay attention to. Here, one more important fact necessary for theoretical description of the process is to be mentioned. If we consider elastic deformations, conditions of consistency (Saint Vincent identities) are relevant for full deformations only. In particular, the system of equations isn't kinematically determined any more.

Moreover, elastic deformations, (with no regard to their values) are principally important to determine the type of the system and to set boundary problems. We confine ourselves with the simplest case, when the body is isotropic and homogenous to elastic deformations. In terms of Hook's law,

$$\Delta\sigma_{ij} = \lambda \cdot (\Delta\varepsilon_{11}^e + \Delta\varepsilon_{22}^e + \Delta\varepsilon_{33}^e) \delta_{ij} + 2\mu\Delta\varepsilon_{ij}^e. \quad (13)$$

Here and later $\Delta\varepsilon_{ij}^e$ is increment of elastic deformations, λ, μ are elastic Lamé parameters. Increments of full deformations are made of two components:

$$\Delta\varepsilon = \Delta\varepsilon_{ij}^e + \Delta\varepsilon_{ij}^p$$

Here

$$\frac{1}{2} \left(\frac{\partial \Delta u_i}{\partial x_j} + \frac{\partial \Delta u_j}{\partial x_i} \right) = \Delta\varepsilon_{ij}^e + \Delta\varepsilon_{ij}^p. \quad (14)$$

Therefore, the closed model of rigid plastic body is enriched by 6 new unknowns $\Delta\varepsilon_{ij}^e$ and 6 new equations (13). 6 equations (12) were replaced by 6 equations (14). Therefore, equations and unknowns are balanced – the system of equations for elastoplastic body is closed.

Conclusions

1. Implementation of the new integral characteristic of shearing stresses results in the condition of plasticity, which is proportional to the relation of the octahedral shearing stress square to the maximal shearing stress.
2. There are characteristic surfaces with normals in a rigidly plastic variant of the model, which are in line with the main stresses direction.
3. In general, the accepted condition of plasticity is the reason for a closed consistent model, which can be used to solve various boundary problems.

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