On the Eulerian recurrent lengths of complete bipartite graphs and complete graphs

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Abstract. An Eulerian circuit of a graph is a circuit that contains all of the edges of the graph. A graph that has an Eulerian circuit is called an Eulerian graph. The Eulerian recurrent length of an Eulerian graph G is the maximum of the length of a shortest subcycle of an Eulerian circuit of G. In other words, if every Eulerian circuit of an Eulerian graph G has a subcycle of length less than or equal to l, and there is an Eulerian circuit of G that has no subcycle of length less than l, then the Eulerian recurrent length of G is l. The Eulerian recurrent length of graph G is abbreviated to the ERL of G, and denoted by ERL(G). In this paper, the ERL's of complete bipartite graphs are given. Let m and n be positive even integers with $m \geq n$. It is shown that $\mathrm{ERL}(K_{m,n}) = 2n - 4$ if $n = m \geq 4$, and $\mathrm{ERL}(K_{m,n}) = 2n$ otherwise. Furthermore, upper and lower bounds on the ERL's of complete graphs are given. It is shown that $n - 4 \leq \mathrm{ERL}(K_n) \leq n - 2$ holds for every odd integer n greater than or equal to 7.

1. Introduction

An Eulerian circuit of a graph is a circuit that contains all of the edges of the graph. A graph that has an Eulerian circuit is called an Eulerian graph. Finding an Eulerian circuit of a graph, that is to say drawing the graph with a single stroke of the brush, is a fundamental problem that was studied by Euler in the dawn of graph theory. It is known that a connected graph is Eulerian if and only if every degree of a vertex of the graph is even, and there is a linear time algorithm to find an Eulerian circuit of an Eulerian graph. Algorithms to find an Eulerian circuit are utilized for reconstructing original long base sequences from short fragments of DNA in the field of bioinformatics[1].

We investigate a problem finding an Eulerian circuit such that the length of a shortest subcycle of the Eulerian circuit is as long as possible. We call the problem the Eulerian recurrent length problem (ERLP). The Eulerian recurrent length of an Eulerian graph G is the maximum of the length of a shortest subcycle of an Eulerian circuit of G. In other words, if every Eulerian circuit of an Eulerian graph G has a subcycle of length less than or equal to l, and there is an Eulerian circuit of G that has no subcycle of length less than l, then the Eulerian recurrent length of G is l. The Eulerian recurrent length of graph G is abbreviated to the ERL of G, and denoted by ERL(G). For example, the ERL of the graph in figure 1 that consists of $3n$ vertices is n. We hope that finding the ERL of a graph is useful for solving some optimization problem.

It has been proved that there is no approximation algorithm with a constant approximation ratio for the ERLP[2]. In this paper, the ERL's of complete bipartite graphs are given. Let m and n be integers with $0 < n \leq m$. It is shown that $\mathrm{ERL}(K_{m,n}) = 4n - 4$ if $n = m$, and

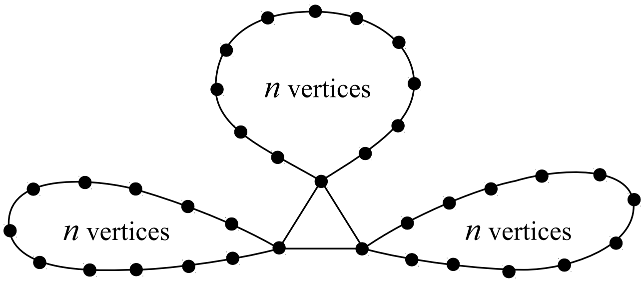

Figure 1. A graph whose ERL equals n.

$\mathrm{ERL}(K_{m,n}) = 4n$ otherwise. Those results are included in our articles[3, 4] written in Japanese. Furthermore, an upper and lower bound on the ERL's of complete graphs are given. It is shown that $n - 4 \leq \mathrm{ERL}(K_n) \leq n - 2$ holds for every odd integer n greater than or equal to 7. The lower bound slightly improves the previous one[3]. As shown above, the ERL's of complete bipartite graphs and complete graphs are close to trivial upper bounds.

In the next section, we shall define several notions necessary for the arguments that follow. In section 3, we shall give the ERL's of complete bipartite graphs. In section 4, we shall give an upper and lower bound on the ERL's of complete graphs. In the last section, we shall give conclusions and remarks about a further challenge to determine the ERL's of complete graphs.

2. Preliminaries

A walk is an alternating sequence of vertices and edges $v_0 \to e_1 \to v_1 \to \cdots \to v_{k-1} \to e_k \to v_k$, beginning and ending with a vertex, such that, for each $i \in \{1, 2, \ldots, k\}$, v_{i-1} and v_i are both end vertices of e_i, and v_{i-1} is different from v_i if e_i is not a loop. Every graph that appears in this paper is a simple undirected graph. We may, therefore, express a walk W with only its vertices as $v_0 \to v_1 \to \cdots \to v_m$, where v_0 is the initial vertex and v_m the terminal vertex. The walk W is said to be a v_0-v_m walk, or a walk from v_0 to v_m. A walk is said to be closed if the initial and final vertex are identical. If a walk is closed, then the walk is expressed as $W = v_0 \to v_1 \to \cdots \to v_m \to v_0$. In this case, the walk W can be written as $v_i \to v_{i+1} \to \cdots \to v_m \to v_0 \to \cdots v_i$ for each $i = 0, 1, \ldots, m$, since each vertex in the walk can be regarded as the initial and end vertex. The length of a walk is the number of edges in the walk, even if the walk is closed.

A trail is a walk such that all its edges are distinct. A circuit is a closed trail. A path is a trail such that all its vertices are distinct except that its initial and final vertex are identical. A cycle is a closed path that has at least one edge. Let G be a graph, and W_1 and W_2 walks in G. If W_1 is a subsequence of W_2, then W_1 is said to be a subwalk of W_2. If C is a cycle and a subwalk of W, then C is said to be a subcycle of W. The terms, subtrail, subcircuit, and subpath, are also defined in the same manner. We may regard a path of a graph as the subgraph induced from the vertices in the path.

An Eulerian circuit of a graph G is a circuit of G that contains all of the edges of G. A graph is Eulerian if it has an Eulerian circuit. It is a well-known fact that a graph is Eulerian if and only if the degree of each vertex of the graph is even.

3. The Eulerian recurrent length of complete bipartite graphs

The ERL's of complete bipartite graphs are given in the following theorem.

Theorem 1 *Let m and n be positive even integers with $m \geq n$. If $m = n \geq 4$, then $\mathrm{ERL}(K_{m,n}) = 2n - 4$ holds. Otherwise, $\mathrm{ERL}(K_{m,n}) = 2n$ holds.*

Proof. It is clear that $\mathrm{ERL}(K_{2,2}) = 4$ holds. The proof immediately follows from this fact and two lemmas below. □

The following lemma and the proof are both presented in [3].

Lemma 1 *Let n be an even integer greater than or equal to 4. Then,* $\mathrm{ERL}(K_{n,n}) = 2n - 4$ *holds.*

Proof. First, we construct an Eulerian circuit T of $K_{n,n}$ such that the length of a shortest subcycle of T is at least $2n-4$. Without loss of generality, we may assume that the vertex-set of $K_{n,n}$ is $\{0, 1, \ldots, 2n-1\}$ and that all of the edges of $K_{n,n}$ join a vertex in $U = \{0, 2, 4, \ldots, 2n-2\}$ and a vertex in $V = \{1, 3, 5, \ldots, 2n-1\}$. For each even integer $k = 0, 2, \ldots, n-2$, let H_k denote the Hamilton path of $K_{n,n}$ defined as

$$\begin{aligned} H_k &= 0 \to (2k+1) \bmod 2n \to 2 \to (2k+3) \bmod 2n \to \cdots \to 2n-2 \\ &\quad \to (2k+2n-1) \bmod 2n \, . \end{aligned}$$

The edge-set of $K_{n,n}$ is partitioned into $n/2$ Hamilton cycles C_0, C_2, …, and C_{n-2}, where each C_k is expressed in the form of $H_k \to 0$. It follows easily from the definition of H_k's that the Hamilton cycles C_0, C_2, …, and C_{n-2} are edge-disjoint each other. Therefore, the closed walk

$$T = H_0 \to H_2 \to H_4 \to \cdots \to H_{n-2} \to 0$$

is an Eulerian circuit.

Since T is a concatenation of Hamilton cycles, every shortest subcycle of T must be contained in a trail $H_k \to H_{(k+2) \bmod n}$ for some $k \in \{0, 2, \ldots, n-2\}$. We assume that k is an even integer with $0 \le k \le n-2$, and shall show that the length of any subcircuit τ of $H_k \to H_{(k+2) \bmod n}$ is not less than $2n-4$. It follows easily from the definition of H_k's that if i is an even integer with $0 \le i \le n-1$, then the length of the subcircuit of $H_k \to H_{(k+2) \bmod n}$ from the vertex i in H_k to the vertex i in $H_{(k+2) \bmod n}$ is equal to $2n$. We may, therefore, assume that there is an odd integer j such that τ is the subcircuit from j in H_k to j in $H_{(k+2) \bmod n}$.

Let x and y denote the integers defined by

$$H_k = 0 \to \cdots \to x \to j \to \cdots \to (2k+2n-1) \bmod 2n$$

and

$$H_{(k+2) \bmod n} = 0 \to \cdots \to y \to j \to \cdots \to (2k+2n+3) \bmod 2n \, .$$

Then, it follows immediately from the definition of H_k's that $x = (j-2k-1) \bmod 2n$ and $y = (j-2k-5) \bmod 2n$. Therefore, the length of the latter part of H_k from j is $l = 2n - ((j-2k-1) \bmod 2n) - 2$, and the length of the former part of $H_{(k+2) \bmod n}$ to j is $r = ((j-2k-5) \bmod 2n) + 1$. It follows from $n \ge 4$ that if $(j-2k-1) \bmod 2n > (j-2k-5) \bmod 2n$ then $(j-2k-1) \bmod 2n = ((j-2k-5) \bmod 2n) + 4$. We, therefore, have $l + r + 1 \ge 2n-4$. Thus, it is concluded that the length of a shortest subcycle of T is not less than $2n-4$.

Next, we assume that there is an Eulerian circuit T of $K_{n,n}$ that has no subcycle of length less than or equal to $2n-4$, and shall derive a contradiction. Since the order of $K_{n,n}$ is $2n$, T has a subcycle of length at most $2n$. Suppose that T has a subcycle

$$S = s_1 \to s_2 \to \cdots \to s_{2n} \to s_1$$

of length $2n$. Since any bipartite graph have no cycles of odd length, $E = \{s_2, s_4, \ldots, s_{2n}\}$ and $O = \{s_1, s_3, \ldots, s_{2n-1}\}$ are the vertex classes of $K_{n,n}$, in other words, either $E = U$ and $O = V$ or $E = V$ and $O = U$ holds. Let x_1 and x_2 denote the vertices defined by

$$T = \cdots \to S \to x_1 \to x_2 \to \cdots \, .$$

It follows from the definition that $x_1 \in E$ and $x_2 \in O$. Any subtrail of T must satisfy the following two conditions.

Table 1. Violations of the conditions when $x_1 \neq s_4$

z	Walk $S \to z$ violates ...
s_2, s_{2n}	Condition 1
$s_6, s_8, \ldots, s_{2n-2}$	Condition 2

Table 2. Violations of the conditions when $x_1 = s_4$

z	Walk $S \to s_4 \to z$ violates ...
s_1, s_3, s_5	Condition 1
$s_7, s_9, \ldots, s_{2n-1}$	Condition 2

Table 3. Violations of the conditions when $x_1 \neq s_{2n}$

z	Walk $S \to z$ violates ...
s_2, s_{2n-2}	Condition 1
$s_4, s_6, \ldots, s_{2n-4}$	Condition 2

Condition 1 No edge appears more than two times in the subtrail.
Condition 2 There is no subcycle of the subtrail of length less than or equal to $2n - 4$.

From table 1, we have $x_1 = s_4$. Furthermore, from table tab02, we conclude that $S \to x_1 \to x_2$ violates Condition 1 or Condition 2 for any x_1 and x_2.

Therefore, we may assume that any subtrail of T satisfies Condition 3 along with Condition 1 and 2 until the end of this proof.

Condition 3 There is no subcycle of the subtrail of length exactly equal to $2n$.

Now, suppose that T has a subcycle

$$S = s_1 \to s_2 \to \cdots \to s_{2n-2} \to s_1$$

of length $2n - 2$. We may express the vertex classes of $K_{n,n}$ as $E = \{s_2, s_4, \ldots, s_{2n}\}$ and $O = \{s_1, s_3, \ldots, s_{2n-1}\}$, where s_{2n-1} and s_{2n} denote the vertices that S does not include. Let x_1, x_2, x_3, and x_4 denote the vertices defined by

$$T = \cdots \to S \to x_1 \to x_2 \to x_3 \to x_4 \to \cdots .$$

By definition, we have $\{x_1, x_3\} \subseteq E$ and $\{x_2, x_4\} \subseteq O$. Then, $x_1 = s_{2n}$ follows from table 3. Then, $x_2 \in \{s_3, s_{2n-1}\}$ follows from table 4. Since $x_2 \neq s_3$ follows from table 5, we have $x_2 = s_{2n-1}$. Furthermore, $x_3 = s_4$ follows from table 6. From table 7, we conclude that $S \to x_1 \to x_2 \to x_3 \to x_4$ violates Condition 1, 2 or 3 for any x_4, a contradiction derived.

We have thus proved the theorem.

□

Table 4. Violations of the conditions when $x_1 = s_{2n}$ and $x_2 \notin \{s_3, s_{2n-1}\}$

z	Walk $S \to s_{2n} \to z$ violates ...
s_1	Condition 1
$s_5, s_7, \dots, s_{2n-3}$	Condition 2

Table 5. Violations of the conditions when $x_1 = s_{2n}$ and $x_2 = s_3$

z	Walk $S \to s_{2n} \to s_3 \to z$ violates ...
s_2, s_4, s_{2n}	Condition 1
$s_6, s_8, \dots, s_{2n-2}$	Condition 2

Table 6. Violations of the conditions when $x_1 = s_{2n}$, $x_2 = s_{2n-1}$, and $x_3 \neq s_4$

z	Walk $S \to s_{2n} \to s_{2n-1} \to z$ violates ...
s_{2n}	Condition 1
$s_6, s_8, \dots, s_{2n-2}$	Condition 2
s_2	Condition 3

Table 7. Violations of the conditions $x_1 = s_{2n}$, $x_2 = s_{2n-1}$, and $x_3 = s_4$

z	Walk $S \to s_{2n} \to s_{2n-1} \to s_4 \to z$ violates ...
s_3, s_5, s_{2n-1}	Condition 1
s_1	Condition 2
$s_7, s_9, \dots, s_{2n-3}$	Condition 2

The following lemma and proof are both presented in our article[4]. The proof in the previous article, however, includes many errors.

Lemma 2 *Let m and n be positive integers with $n < m$. Then,* $\mathrm{ERL}(K_{2m,2n}) = 4n$ *holds.*

Proof. Let $A = \{(0, x) \mid 0 \leq x \leq 2n - 1\}$ and $B = \{(1, x) \mid 0 \leq x \leq 2m - 1\}$ be the vertex classes of the complete bipartite graph $K_{2m,2n}$. Let k denote positive integer $\gcd(m, n) = \gcd(2m, 2n)/2$. For each $j \in \{0, 1, \dots, k - 1\}$, we define trail H_j and circuit C_j of G as follows:

$$\begin{aligned} H_j &= (0,0) \to (1,2j) \to (0,1) \to (1,(2j+1) \bmod 2m) \to \cdots \to \\ &\quad (0, i \bmod 2n) \to (1,(2j+i) \bmod 2m) \to \cdots \to (0,2n-1) \to (1,(2j-1) \bmod 2m), \\ C_j &= H_j \to (0,0). \end{aligned}$$

That is to say, $(0, 0)$ is the initial vertex of H_j and C_j, and for every $i \in \{0, 1, 2, \ldots, 2n-1\}$, $(0, i \bmod 2n)$ and $(1, (2j+i) \bmod 2m)$ are $2i$ and $2i+1$ edges distant from $(0, 0)$ on C_j, respectively. Furthermore, the length of C_j is the double of the minimum of positive integer i that satisfies both

$$i \equiv 0 \pmod{2n} \quad \text{and} \quad 2j + i \equiv 2j \pmod{2m} \,. \tag{1}$$

Condition (1) is equivalent to the following:

$$i \text{ is a multiple of } 2k, \quad i/(2k) \equiv 0 \pmod{n/k}, \quad \text{and} \quad i/(2k) \equiv 0 \pmod{m/k} \,,$$

and, by the Chinese remainder theorem, there is a unique integer i that satisfies the condition above and $0 < i/(2k) \leq nm/k^2$. Since $i = 2nm/k$ satisfies condition (1), the length of C_j is $4nm/k$ for every j.

For any integers i and j with

$$0 \leq i \leq k-1, \quad 0 \leq j \leq k-1, \quad \text{and} \quad i \neq j, \tag{2}$$

there is no edge e such that $e \in E(C_i) \cap E(C_j)$, where $E(C_i)$ and $E(C_j)$ denote the set of all the edges on C_i and C_j, respectively. It is for the reason that a contradiction follows from the existence of such an edge e as follows. Assume that there is such an edge e. Then, there must be integers p and q such that one of the following three conditions holds:

(a) $(0, p \bmod 2n) = (0, q \bmod 2n)$ and $(1, (2i+p) \bmod 2m) = (1, (2j+q) \bmod 2m)$.
(b) $(0, p \bmod 2n) = (0, q \bmod 2n)$ and $(1, (2i+p) \bmod 2m) = (1, (2j+q-1) \bmod 2m)$.
(c) $(0, p \bmod 2n) = (0, q \bmod 2n)$ and $(1, (2i+p-1) \bmod 2m) = (1, (2j+q-1) \bmod 2m)$.

It is impossible that condition (b) holds since $p \bmod 2n = q \bmod 2n$ implies $p \equiv q \pmod 2$, and $(2i+p) \bmod 2m = (2j+q-1) \bmod 2m$ implies $p \not\equiv q \pmod 2$. It is also impossible that condition (a) or (b) holds. If condition (a) or (b) holds, then both $p - q \equiv 0 \pmod{2k}$ and $i - j \equiv 0 \pmod k$ hold. Then, $i - j \equiv 0 \pmod k$ contradicts condition (2). Hence, the circuit

$$T = H_0 \to H_1 \to \cdots \to H_{k-1} \to 0$$

obtained by connecting the circuits C_0, C_1, $\ldots$, C_{k-1} in this order is an Eulerian circuit of $K_{2m,2n}$.

We now can readily verify the following two facts. Each vertex in A and B appears at regular $4n$ and $4m$ edges intervals, respectively, on C_j for each $j \in \{0, 1, 2, \ldots, k-1\}$. Furthermore, if a vertex v appears at position p in trail H_j and q in trail $H_{(j+1) \bmod k}$, then p and q are at least $4(m-1) \geq 4n$ edges distant each other. Thus, we conclude the proof. $\square$

4. The Eulerian recurrent lengths of complete graphs

We give an upper and lower bound on the ERL of complete graphs K_n that consists of odd number of vertices in this section.

4.1. An upper bound on the ERL's of complete graphs

We give an upper bound on the ERL's of complete graphs K_n for odd integers n greater than or equal to 5 as follows:

$$\mathrm{ERL}(K_n) \leq n - 2 \,,$$

which immediately follows from the following theorem. The theorem and the proof are both presented in [3].

Table 8. Violations of the conditions $x_1 = s_{2n}$, $x_2 = s_{2n-1}$, and $x_3 = s_4$

z	Walk $S \to z$ violates ...
s_1	Condition 4
s_2, s_n	Condition 5
$s_4, s_5, \ldots, s_{n-1}$	Condition 6

Table 9. Violations of the conditions $x_1 = s_{2n}$, $x_2 = s_{2n-1}$, and $x_3 = s_4$

z	Walk $S \to s_3 \to z$ violates ...
s_3	Condition 4
s_1, s_2, s_4	Condition 5
$s_5, s_6, \ldots, s_n$	Condition 6

Theorem 2 *Let n be an odd integer with $n \geq 5$. Then, every Eulerian circuit of K_n has a subcycle of length at most $n-2$.*

Proof. The strategy for the proof is similar to that of Theorem 1. Let T be an Eulerian circuit of K_n. We derive a contradiction from the assumption that the length of a shortest subcycle of T is greater than $n-2$, proving that T always has a subcycle of length at most $n-2$. Any subtrail S of T always satisfies the following three conditions.

Condition 4 S has no loops, where a loop is an edge joining a vertex to itself.

Condition 5 For any pair of vertices (v, w), S does not have two or more edges joining v and w

Condition 6 S has no subcycles of length at most $n-2$.

Since the order of K_n is n, T has a subcycle S of length at most n. First, suppose that T has has a subcycle

$$S = s_1 \to s_2 \to \cdots \to s_n \to s_1$$

of length n. Let x_1 and x_2 be the vertices such that

$$T = \cdots \to S \to x_1 \to x_2 \to \cdots .$$

Then, $x_1 = s_3$ follows from table 8. Furthermore, it follows from table 9 that $S \to x_1 \to x_2$ violates Condition 4, Condition 5 or Condition 6 for any x_2. Thus, we obtain a contradiction.

Next, suppose that T has a subcycle

$$S = s_1 \to s_2 \to \cdots \to s_{n-1} \to s_1$$

of length exactly $n-1$. Let s_n denote the unique vertex not contained in S, and x_1, x_2, and x_3 be the vertices such that

$$T = \cdots \to S \to x_1 \to x_2 \to x_3 \to \cdots .$$

Then, we may assume that any subtrail S of T satisfies Condition 7 along with Condition 4, 5 and 6 until the end of this proof.

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Table 10. Violations of the conditions $x_1 = s_{2n}$, $x_2 = s_{2n-1}$, and $x_3 = s_4$

z	Walk $S \to z$ violates ...
s_1	Condition 4
s_2, s_{n-1}	Condition 5
$s_3, s_4, \ldots, s_{n-2}$	Condition 6

Table 11. Violations of the conditions $x_1 = s_{2n}$, $x_2 = s_{2n-1}$, and $x_3 = s_4$

z	Walk $S \to s_n \to z$ violates ...
s_n	Condition 4
s_1	Condition 5
$s_4, s_5, \ldots, s_{n-1}$	Condition 6
s_2	Condition 7

Table 12. Violations of the conditions $x_1 = s_{2n}$, $x_2 = s_{2n-1}$, and $x_3 = s_4$

z	Walk $S \to s_n \to s_3 \to z$ violates ...
s_3	Condition 4
s_2, s_4, s_n	Condition 5
s_1	Condition 6
$s_5, s_6, \ldots, s_{n-1}$	Condition 6

Condition 7 There is no subcycle of S of length n.

Then, $x_1 = s_n$ follows from table 10. Furthermore, $x_2 = s_3$ follows from table 11. From table 12, we conclude that $S \to x_1 \to x_2 \to x_3$ violates Condition 4, 5, 6 or 7 for any x_1, x_2, and x_3.

We have thus proved the theorem.

□

4.2. A lower bound on the ERL's of complete graphs

We give a lower bound of the ERL of complete graphs K_n for odd integers n greater than or equal to 7:

$$\mathrm{ERL}(K_n) \geq n - 4 \ ,$$

which immediately follows from the following theorem stated in our recent work[5]. We obtain the theorem by slightly improving our previous result in [3].

Theorem 3 *Let n be an odd integer with $n \geq 7$. Then, there is an Eulerian circuit C of K_n such that the length of any subcycle of C is greater than or equal to $n - 4$.*

Proof. Assume that the vertex set of complete graph K_n that consists of n vertices is $\{0, 1, 2, \ldots, n-1\}$. Let H_k denote the Hamiltonian path $n-1 \to v_0(k) \to v_1(k) \to \cdots \to v_{n-2}(k)$

of K_n for each $k \in \{0, 1, 2, \ldots, n-2\}$, where $v_i(k)$ is defined recursively as follows:

$$v_i(k) = \begin{cases} k & \text{if } i = 0, \\ (v_{i-1}(k) + i) \bmod (n-1) & \text{if } i > 0 \text{ and } i \text{ is odd}, \\ (v_{i-1}(k) - i) \bmod (n-1) & \text{otherwise.} \end{cases}$$

It is known that every complete graph K_n consisting of odd number k of vertices is decomposed into $(n-1)/2$ Hamiltonian cycles H_0, H_1, ..., $H_{(n-3)/2}$[6]. Figure 2 depicts H_0, H_1, and H_2 for complete graph K_9 consisting of 9 vertices.

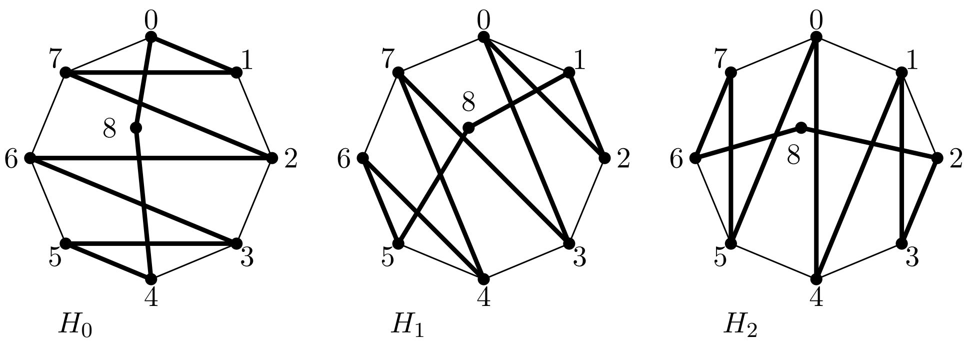

Figure 2. $H_0 \to n-1$, $H_1 \to n-1$, and $H_2 \to n-1$ for K_9.

Let the Eulerian circuit C_n of K_n be defined as

$$C_n = \begin{cases} \begin{array}{c} H_0 \to H_2 \to \cdots \to H_{(n-5)/2} \to H_{(n-3)/2} \\ \to H_{(n-3)/2-2} \to \cdots \to H_1 \to n-1 \end{array} & \text{if } n \equiv 1 \pmod 4, \\ \begin{array}{c} H_0 \to H_2 \to \cdots \to H_{(n-3)/2} \to H_{(n-5)/2} \\ \to H_{(n-5)/2-2} \to \cdots \to H_1 \to n-1 \end{array} & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Then, the following hold.

$$v_i(k) = v_j(k+1) \quad \text{implies} \quad n-2 \le n+j-i \le n+2 \quad \text{for each} \quad k \in \{0, 1, \ldots, (n-5)/2\}\,, \tag{3}$$

$$v_i(k) = v_j(k-1) \quad \text{implies} \quad n-2 \le n+j-i \le n+2 \quad \text{for each} \quad k \in \{1, 2, \ldots, (n-3)/2\}\,, \tag{4}$$

$$v_i(k) = v_j(k+2) \quad \text{implies} \quad n-4 \le n+j-i \le n+4 \quad \text{for each} \quad k \in \{0, 1, \ldots, (n-7)/2\}\,, \tag{5}$$

and

$$v_i(k) = v_j(k-2) \quad \text{implies} \quad n-4 \le n+j-i \le n+4 \quad \text{for each} \quad k \in \{2, 3, \ldots, (n-3)/2\}\,. \tag{6}$$

Thus, it follows from (3), (4), (5), and (6) that the length of any subcycle of C_n is greater than or equal to $n-4$ for any integer n greater than or equal to 7. Thus, we conclude the proof. $\square$

5. Concluding remarks

We have given the exact values of the Eulerian recurrent lengths of complete bipartite graphs in Theorem 1 by gathering our results in two articles. We have then described the proof of those values. We also have given an upper and lower bound on the Eulerian recurrent lengths of complete graphs in Theorems 2 and 3, and have described the proofs. We have obtained the lower bound by slightly improving our previous result.

We have already proved that $\mathrm{ERL}(K_n) \leq n-3$ holds for every odd integer n greater than or equal to 7, in our recent article[5]. It has been verified by computer experiments that $\mathrm{ERL}(K_n) = n-3$ holds for each integer $n \in \{7, 9, 11, 13\}$. I currently conjecture that $\mathrm{ERL}(K_n) = n-4$ holds for every odd integer n greater than or equal to 15. If the conjecture holds, then the Eulerian circuit described in the proof of Theorem 3 is an optimal solution for the ERLP constrained to input only complete graphs that consists of odd number n of vertices with $n \geq 15$.

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