Improved FOCUSS method with two-step reweighted ℓ_2 minimization

X Han[1] and Z Jiang[1]

[1] Air Force Airborne Academy, Guilin, P.R.China

E-mail: thuhxb@gmail.com

Abstract. FOCal Underdetermined System Solver (FOCUSS) is a useful method through reweighted ℓ_2 minimization for sparse recovery. In this paper, we introduce an improved FOCUSS by enhancing sparsity with two reweighted ℓ_2 minimization. The reweighted FOCUSS method has higher mission success rate and better accuracy than FOCUSS. The simulation results illustrate the advantage of reweighted FOCUSS.

1. Introduction

The problems of finding sparse solutions to underdetermined system have been the hot spot because of the increasing applications in many fields, such as signal processing, image processing, information encoding, etc. This problem can be modeled as:

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n} \tag{1}$$

where $\mathbf{y} = (y_1, \cdots, y_m)^T \to \mathbf{R}^m$ is an observable vector, $\mathbf{x} = (x_1, \ldots, x_n)^T \to \mathbf{R}^n$ is an unknown input vector, $\mathbf{A}$ is a known $m \times n$ basis matrix, and $\mathbf{n}$ represents noise vector.

The main objective is to recover $\mathbf{x}$ from (1) such that $\mathbf{x}$ is sparse, $m \times n$, and $\mathbf{A}$ satisfies some property (e.g. RIP [1]).

In the noise-free setup, it has been asserted that exact reconstruction of $\mathbf{x}$ can be guaranteed by solving a convex problem [1]:

$$\min_x \|\mathbf{x}\|_1 \text{ s.t. } \mathbf{y}{=}\mathbf{A}\mathbf{x} .$$

For (1), the solution of approximate reconstruction can be expressed by another convex problem with Lagrange form as

$$\min_x \|y - Ax\|_{\ell_2}^2 + \lambda \|x\|_{\ell_1} .$$

However, the computational burden of convex optimization is very high, and thus unsuitable to large scale problems and practice applications. At the same time, many substitutive algorithms have been proposed for this problem, such as greedy algorithms (e.g., MP, OMP [2], CoSaMP [3], etc.), FOCUSS algorithm(s) [4]. The performance of greedy algorithms can only be guaranteed only if $\mathbf{A}$ satisfies rigorous RIP condition and usually a little worse than others; FOCUSS algorithms are advantageous in terms of computational complexity and loose requirement on $\mathbf{A}$.

In the correspondence, FOCUSS algorithms are worth improving. As mentioned that FOCUSS is an iterative procedure of reweighted ℓ_2 minimization, the main contribution of this paper is to improve FOCUSS with another iterative reweighted procedure.

IOP Conf. Series: Materials Science and Engineering **57** (2014) 012008 doi:10.1088/1757-899X/57/1/012008

2. Drawback analysis of FOCUSS

Table 1. FOCUSS algorithms

basic FOCUSS	regularized FOCUSS
step 1: $\mathbf{W}_k = \mathrm{diag}(\mathbf{x}_{k-1})$ step 2: $\mathbf{q}_k = (\mathbf{AW}_k)^\dagger \mathbf{y}$, step 3: $\mathbf{x}_k = \mathbf{W}_k\mathbf{q}_k$	step 1: $\mathbf{W}_k = \mathrm{diag}(\lvert\mathbf{x}_{k-1}\rvert^{1-p/2})$ step 2: $\mathbf{q}_k = \mathbf{A}_k^H\left(\mathbf{A}_k\mathbf{A}_k^H + \lambda\mathbf{I}\right)^{-1}\mathbf{y}$, *where* $\mathbf{A}_k = \mathbf{AW}_k$ step 3: $\mathbf{x}_k = \mathbf{W}_k\mathbf{q}_k$

The basic form and regularized form of FOCUSS algorithm are described [4] as Table 1. In Table 1, λ is Lagrange parameter [5] and $0 < p \leq 1$ is pre-set norm-factor. FOCUSS algorithm(s) can be considered a reweighted ℓ_2 minimization because in each iteration, it is equivalent to find the optimum solutions to (2) [4,5].

$$\min\|\mathbf{q}_k\|_{\ell_2}^2 \text{, s.t.} \|\mathbf{Ax}-\mathbf{y}\|_{\ell_2} \leq \beta$$
$$\text{where } \|\mathbf{q}_k\|_{\ell_2}^2 = \|\mathbf{Wx}_k\|_{\ell_2}^2 = \sum_{i=1}^{n}\left|\frac{x_{k,i}}{x_{k-1,i}}\right|^2 \tag{2}$$

where β is noise threshold, and $\mathbf{W} = \mathbf{W}_k^{-1}$.

The reweighted method of FOCUSS is accordant to idea of enhancing sparsity by reweighted ℓ_1 minimization [6], which mentions that large entries in w_i force the solution $\mathbf{x}$ to concentrate on the indices where w_i is small in order to approach ℓ_0 minimization further more.

$\mathbf{x}_k$ in Table 1 is equivalent to

$$\mathbf{x}_k = \mathbf{W}_k\left(\mathbf{W}_k^H\mathbf{A}^H\mathbf{A}W_k\right)^{-1}\mathbf{W}_k^H\mathbf{A}^H\mathbf{y}$$
$$\text{or } \mathbf{x}_k = \mathbf{W}_k\left(\mathbf{W}_k^H\mathbf{A}^H\mathbf{A}W_k + \lambda\mathbf{I}\right)^{-1}\mathbf{W}_k^H\mathbf{A}^H\mathbf{y} \tag{3}$$

From (3) we can find that $\mathbf{x}_k$, the approximate solution of $\mathbf{x}$ in k-th iteration depends on the constant entries $\mathbf{A}^H\mathbf{y}$. Let $\mathbf{A} = [\mathbf{a}_1,\cdots,\mathbf{a}_n]$, $\mathbf{T}$ represents the indices set of non-zero entries, $\bar{\mathbf{T}}$ represents the indices set of zero entries. Suppose $t \in \mathbf{T}$ is corresponding to the index of the smallest non-zero entry in $\mathbf{x}$, $\hat{x}_{k,t}$, the estimation of x_t is desponded on $\mathbf{a}_t^H\mathbf{y}$. Due to

$$\mathbf{a}_t^H\mathbf{y} = \mathbf{a}_t^H\mathbf{Ax} + \mathbf{a}_t^H\mathbf{n}, \tag{4}$$

if $\mathbf{a}_t^H\mathbf{Ax}$ and $\mathbf{a}_t^H\mathbf{n}$ have the opposite phase, the power of signal in $\mathbf{a}_t^H\mathbf{y}$ would be crippled even counteracted. So with some possibility, there exists $d \in \bar{\mathbf{T}}$ satisfying $|\hat{x}_{k,t}| < |\hat{x}_{k,d}|$. Under this situation, w_t, which should be smaller than w_d, is bigger than w_d. This situation will never be improved through iterations, and thus leads to the failure of FOCUSS.

3. Reweighted FOCUSS algorithm

In order to overcome the drawback in Section 2, we design the reweighted FOCUSS which utilize another different reweighted ℓ_2 minimization. Similar to [6] and [7], the important thing of reweighted FOCUSS is assigning bigger weights to those elements of $\mathbf{x}$ which are more likely to be zero. Reweighted FOCUSS algorithm is described in Table 2.

In reweighted FOCUSS algorithm, FOCUSS algorithms can be considered the first step of reweighted ℓ_2 minimization and the following procedures can be considered the second step which is equivalent to find the optimum solutions to the following:

Table 2. Reweighted FOCUSS algorithm

Algorithm *reweighted FOCUSS:*

1. Get the initial estimation of $\mathbf{x}_0$ of $\mathbf{x}$ through FOCUSS algorithms;
2. Find the index set $\mathbf{K} \subset \{1, 2, \cdots, n\}$ which corresponds to the largest $C \cdot \dfrac{m}{\log(n/m)}$ elements of $\mathbf{x}_k$ in amplitudes [8], where C is a constant with the numerical value given in [1];
3. Set diagonal matrix Ω with diagonal elements satisfying

$$\begin{cases} \omega_i = 1, i \in \mathbf{K} \\ \omega_i > 1, i \in \bar{\mathbf{K}} \end{cases}$$

and calculate

$$\mathbf{p}_k = \left(\mathbf{A}\Omega\right)^{\dagger} \mathbf{y} \ \text{ or } \mathbf{p}_k = \Omega^H \Omega^H \left(\mathbf{A}\Omega\Omega^H \mathbf{A}^H + \lambda \mathbf{I}\right)^{-1} \mathbf{y},$$
$$\mathbf{x}_k = \Omega \mathbf{p}_k;$$

4. Terminate on convergence or when k attains a specified maximum number of iterations $k_{\max}$; otherwise, increase k and go to step 2.

$$\begin{aligned} &\min \|\mathbf{p}_k\|_{\ell_2}^2, \text{ s.t. } \|\mathbf{A}\mathbf{x} - \mathbf{y}\|_{\ell_2} \le \beta \\ &\text{where } \|\mathbf{p}_k\|_{\ell_2}^2 = \|\mathbf{W}\mathbf{x}_k\|_{\ell_2}^2 = \sum_{i \in \mathbf{K}} |x_{k,i}|^2 + \sum_{i \in \bar{\mathbf{K}}} |\omega_i x_{k,i}|^2 \end{aligned} \tag{5}$$

The method choosing $\mathbf{W}$ in (5) is proved to be efficient to recover $\mathbf{x}$ from (1) in [1]. However, differently from [1] which needs solving convex problem, this paper employs iterated method to solve the reweighted ℓ_2 minimization approximately, which deduces the computational burden greatly.

Through re-assigning the weighted coefficient, if some indices of small nonzero entries in $\mathbf{x}$ are undetected or some indices of zero entries are false-drop under FOCUSS algorithms just like the analysis in Section 2, they can be corrected.

4. Simulation results

This section shows the advantages of recovery ability of reweighted FOCUSS algorithms with numerical simulations, compared to FOCUSS algorithms. In a Monte Carlo simulation, 1000 trials are carried out independently. In each trial, $\mathbf{A}$ is chosen as Gaussian random matrix, entries of which are independently, identically and normally distributed; $\mathbf{x}$ is a s-sparse vector which is input randomly, with normalizing power, i.e., $\sum_i |x_i|^2 = 1$; entries of $\mathbf{n}$ are independently and identically Gaussian distributed with mean zero and variance σ^2; The indices of nonzero coordinate set $\mathbf{T}$ are chosen randomly from a discrete uniform distribution of [1,n]. Then SNR of system can be represented as $1/\sigma^2$. The algorithm in simulations is considered to be successful if all nonzero-locations of $\mathbf{x}$ are found exactly; otherwise the algorithm is considered to be failed.

The algorithms mentioned in the simulation are: FOCUSS, regularized FOCUSS, reweighted FOCUSS and reweighted regularized FOCUSS. Figure1 shows the statistical results of mission success rate, and Figure 2 shows the statistical root-mean-square error (RMSE) curve of signal amplitude recovery when algorithms can find the nonzero-coordinate $\mathbf{T}$ correctly under different SNR scenes. It can be seen from Figure 1 and Figure 2 that reweighted FOCUSS works more robustly and more accurately than FOCUSS, and so does reweighted regularized FOCUSS better than regularized FOCUSS. In Figure 1 and Figure 2, $m=32$, $n=128$, $s=3$, $p=0.5$.

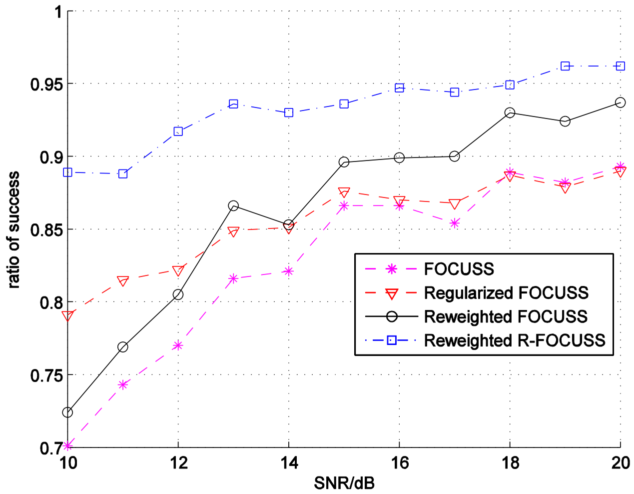

Figure 1. Mission success rate of algorithms in finding the support set correctly.

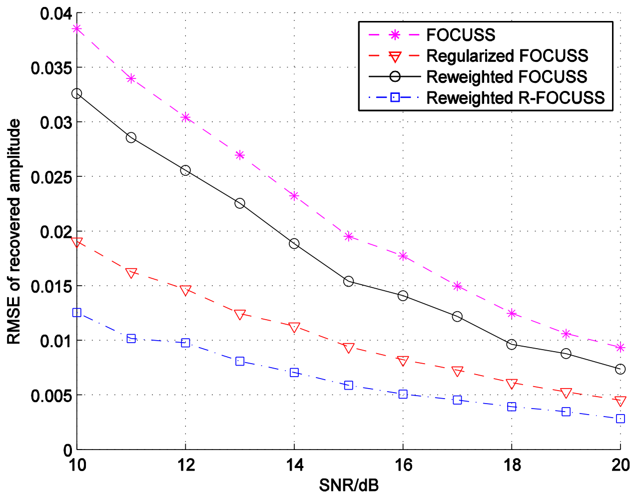

Figure 2. RMSE of signal amplitude recovery.

5. Conclusion

In this paper, for solving sparse-recovery problems, we have analyzed the major drawback of traditional FOCUSS algorithms, and developed the reweighted FOCUSS to approximate sparse solutions. We give the experimental evidence that reweighted FOCUSS can reconstruct the sparse inputs more robustly and accurately than untreated FOCUSS. This reweighted method has never conflict with sparse-recovery algorithms and can be regarded as the post-processing of them, and is easier to apply than reweighted ℓ_1 minimization.

References

[1] Candès E and Tao T 2005 Decoding by linear programming *IEEE Trans. Inf. Theory* **51** 4203
[2] Tropp J and Gilbert A 2007 *IEEE Trans. Inf. Theory* **53** 4655

[3] Needell D and Tropp J 2009 *Appl. Comput. Harmon. Anal.* **26** 301
[4] Gorodnitsky I and Rao B 1997 *IEEE Trans. Signal Process.* **45** 600
[5] Rao B, Engan K, Cotter S and Palmer J 2003 *IEEE Trans. Signal Process.* **51** 760
[6] Candès E, Wakin M and Boyd S 2008 *J. Fourier Anal Appl.* **14** 877
[7] Xu W, Khajehnejad M, Avestimehr A and Hassibi B 2010 *ICASSP* 5498
[8] Candès E, Romberg J and Tao T 2006 *Commun. Pure Appl. Math*. **59** 1207