A design method with differentiability for optimal control problems with non-differential Hamilton-Jacobi equations

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Abstract. As far as optimal feedback controllers for nonlinear systems are concerned, Hamilton-Jacobi(HJ) equations play a key role in the design process. However, we sometimes meet the HJ equations in non-differential situations, which are extremely difficult to solve analytically/numerically. This paper proposes a simple design method to attack such optimal control problems using differentiable solutions of the associated HJ equations. By simulation, we demonstrate the proposed method useful in a non-differentiable situation.

1. Introduction

As far as optimal feedback controllers for nonlinear systems are concerned, Hamilton-Jacobi(HJ) equations play a key role in the design process. However, we sometimes meet the HJ equations in non-differential situations, such as unicycle or car-like vehicle problems, underwater control problems, bilinear dynamical system problems, and so on. Unfortunately, those HJ equations are extremely difficult to solve analytically/numerically. A lot of researchers have been tackling such non-differentiable control problems. However, no powerful technique has been found so far. Note that, especially in the non-differentiable HJ equations, viscosity solutions [1] are generally used instead of differentiable solutions. As far as viscosity solutions are concerned, no effective computational tool exists so far in solving non-differentiable HJ equations.

This paper proposes a simple design method to attack the optimal control problems with non-differentiable HJ equations, based on two steps. The first step is to find non-differentiable points in the state space (say, Algorithm I). This is not an easy task. We propose a unique idea to find such points in an easy fashion. The second step is to divide a whole state space into subspaces, based on the result given by Algorithm I, where non-differentiable points are not contained. Now that we are in a differentiable situation, we apply the existing HJ solvers for each subspace.

The paper is organized as follows. The optimal control problem in question is formulated in Section 2. In Section 3, the simple but effective design method is proposed with differentiability assumptions. Section 4 demonstrates the usefulness of the proposed method. Section 5 gives the conclusion.

IOP Conf. Series: Materials Science and Engineering **83** (2015) 012009 doi:10.1088/1757-899X/83/1/012009

2. Problem formulation

Consider the following dynamical system (1), with the performance index (2) to be minimized.

$$\dot{x} = f(x) + g(x)u \tag{1}$$

$$J = \int_0^\infty (l(x) + u^{\mathrm{T}} R u) dt \tag{2}$$

where, $x \in \mathbb{R}^n$ is a state vector, and $u \in \mathbb{R}^m$ is an input variable. f and g are smooth functions with $f(0) = 0$.

Even with a standard type of optimal control problems, we sometimes meet the non-differentiable HJ equations in deriving feedback controllers. Of course, there are some approaches to such non-differentiable situations. However, we propose a new technique which is completely different from the existing methods to attack such problems. What we are going to do is to select the non-differentiable points in the state space, and take away those points in order to apply the existing differentiable methods. How can we find these non-differentiable points? This is a major issue that we have to solve. For this reason, we give some preliminaries. In the sequel, we use "Distribution" and "Tangent space". Those are given as follows.

Tangent space: Given a manifold $\mathcal{M}$, construct a subspace at $x \in \mathcal{M}$, using all the tangent vectors of $x \in \mathcal{M}$. The subspace is called "Tangent space", denoted by "$T_x(\mathcal{M})$".

Distribution: Consider a linear variational system, $\delta\dot{x} = A\delta x + \sum_{i=1}^{m} b_i \delta u_i$, which is given around a point (x, u) with a nonlinear system $\dot{x} = f(x, u)$, and construct a subspace using the following vectors, b_1, Ab_1, $\cdots$, $A^{n-1}b_1$, $\cdots$, $b_m Ab_m$, $\cdots$, $A^{n-1}b_m$. The subspace is called "Distribution", denoted by "Span$(b_1, Ab_1, \cdots, A^{n-1}b_1, \cdots, b_m, Ab_m, \cdots, A^{n-1}b_m)$" or "Span$(\Delta(x))$". Here,

$$\Delta(x) = (b_1, Ab_1, \cdots, A^{n-1}b_1, \cdots, b_m, Ab_m, \cdots, A^{n-1}b_m). \tag{3}$$

Roughly speaking, Distribution is in a sense related to a controllability matrix. By using Distribution, we can check the reachability of the linearized variational systems.

3. Design method

First, we describe how to find the non-differentiable points in the state space. Next, based on these non-differentiable points, we propose our design method dividing the whole state space into the subspaces, where the subspaces do not contain the non-differentiable points. Because of that, we can apply existing methods which are developed for differentiable situations.

3.1. Non-differentiable points

We describe an algorithm to find non-differentiable points.

Algorithm I

Step 0: Set $D_1 = \varnothing$ and $D_2 = \varnothing$.

Step 1: Derive the following variational system around the point (x_l, u_l) with the nonlinear system (1).

$$\delta\dot{x} = A_l \delta x + B_l \delta u \tag{4}$$

Step 2: Construct "Distribution" at (x_l, u_l), using the variational system as mentioned in Section 2.

Step 3: Construct "Tangent space" at x_l, using the manifold with $l(x)$ as mentioned in Section 2.

Step 4: If Tangent space in Step 3 contains Distribution in Step 2, update D_1 by letting x_l become an element of D_1, and go to Step 5. Otherwise, go to Step 5.

Step 5: If every point of (x_l, u_l) is selected, go to Step 6. Otherwise go back to Step 1.

Step 6: If D_1 contains no element or the origin only, update D_1 by re-setting $D_1 = \varnothing$ and go to Step 7. Otherwise, go to Step 9.

Step 7: Construct Distribution at (x_l, u_l) as in Step 2. If Distribution is not equivalent to the whole state space, update D_2 by letting x_l become an element of D_2, and go to Step 8. Otherwise, go to Step 8.

Step 8: If every point of (x_l, u_l) is selected, stop. Otherwise, go back to Step 7.

Step 9: Update D_2 by re-setting $D_2 = \mathrm{span}(\Delta(x))$ with $(x_l, u_l) = (0, 0)$.

3.2. Design Method

Now, we have found all of the non-differentiable points. Therefore, we are in a position to propose a design method for nonlinear systems with non-differentiable HJ equations, based on Algorithm I.

Design Method

Step 1: Apply Algorithm I to the nonlinear system (1) with the performance index (2).

Step 2: Choose a practical region Ω for the control design in the state space, in order to construct a feedback controller over the such a region.

Step 3: Based on D_1 and D_2, divide the practical region into subregions which do not contain non-differentiable points in each region.

Step 4: Apply the Galerkin approximation method [2] to each subregion, together with basis functions appropriately chosen.

Step 5: Construct a feedback controller over the whole practical region, based on differentiable solutions of subregions given in Step 4 of the design method.

4. Simulation

Four examples are given to show how Algorithm I works in a control design process. Two more examples are given to demonstrate how effective the proposed design method works. One of them is shown in this section.

Consider the following bilinear system (5) with the performance index (6)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & x_1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{5}$$

$$J = \int_0^\infty (5x_1^2 + 5x_2^2 + u_1^2 + u_2^2)dt \tag{6}$$

where a viscosity solution exists in the associated HJ equation. As mentioned before, it would be a difficult task to obtain viscosity solutions analytically/numerically. We tackle this problem, according to the design procedure given in section 3. Applying Algorithm I, we obtain D_1 and D_2, as follows.

$$D_1 = \{x_l | x_{l,1} = 0\}, \ D_2 = \{x_l | x_{l,2} = 0\} \tag{7}$$

Based on such D_1 and D_2, we attack an associated HJ equation.

$$-\frac{1}{4}\left(\frac{\partial V}{\partial x_1}\right)^2 - \frac{x_1^2}{4}\left(\frac{\partial V}{\partial x_2}\right)^2 + 5x_1^2 + 5x_2^2 = 0 \tag{8}$$

By the Galerkin approximate method with the practical region $\Omega = \{(x_l, u_l) \mid -1.5 \leq x_1 \leq 1.5, -1.5 \leq x_2 \leq 1.5\}$, we obtain the solution in Fig.1, where non-differentiable points can be shown on the line of $x_1 = 0$. A comparison with the optimal solution is given in Fig.2, where the optimal one is given numerically by the reference [3]. The results show that the proposed and optimal ones are sufficiently close from a practical point of view.

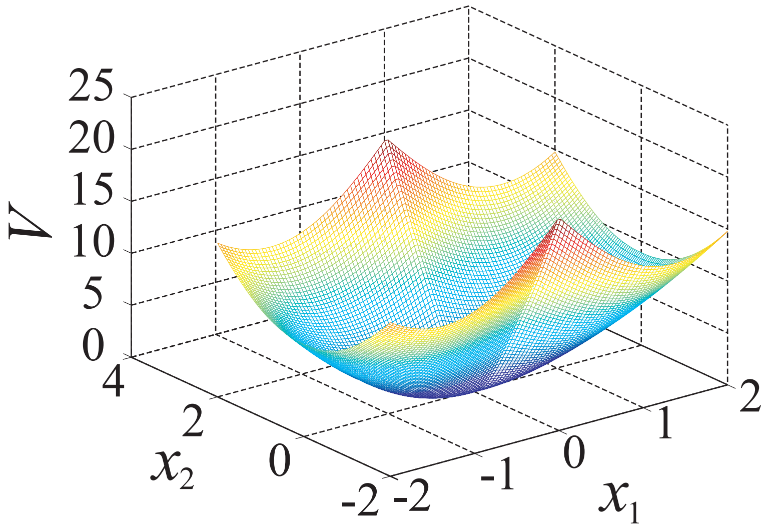

Figure 1. Value of Cost

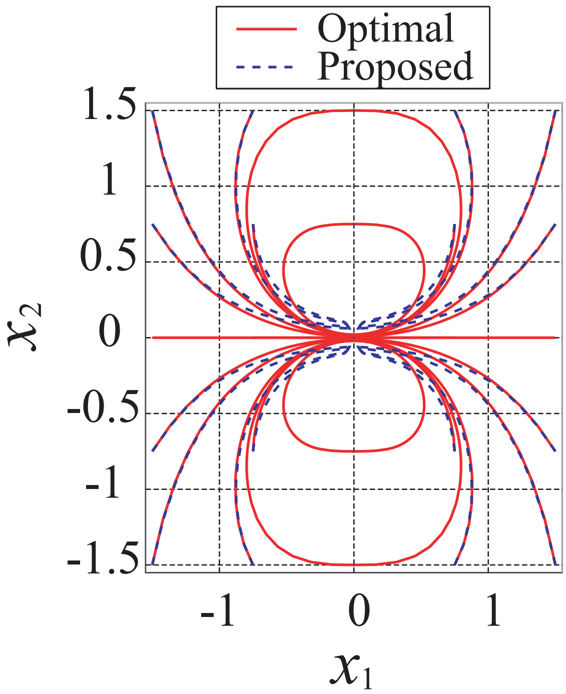

Figure 2. Phase Portrait

5. Conclusion

This paper has proposed an algorithm to find non-differentiable points (Algorithm I), and a design method based on the results given by Algorithm I. Numerical simulation is given in order to demonstrate the usefulness of the proposed method of control design.

This work is supported by JSPS KAKENHI Grant Number 24560551.

References

[1] M. G. Crandall and P. L. Lions: Viscosity Solutions of Hamilton-Jacobi Equations; *American Mathematical Society* (1983)

[2] R. Beard, G. Saridis and J. Wen: Galerkin Approximations of the Generalized Hamilton-Jacobi-Bellman Equation; *Automatica*, Vol. 33, No. 12, pp. 2159-2177 (1997)

[3] J. Imae, R. Torisu: A Riccati-Equation Based Algorithm for Nonlinear Optimal Control Problem; *Proceeding of the 37th IEEE Conference on Decision and Control*, pp. 4422-4427 (1998)