Generalized Calogero system for simple Lie algebra series

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Abstract. We derive equations of motions and solutions for the generalized spin–Calogero models associated for the classical Lie algebra series B_n, C_n BC_n, and D_n using the pole expansion.

1. Generalized Calogero system

The generalized Calogero systems have been introduced in [3]. Corresponding Hamiltonian has the form:

$$H = \frac{1}{2m}\sum_{i=1}^{N} p_i^2 + \frac{1}{2}\alpha_1^2 \sum_{i\neq j}^{N} \left(V(x_i - x_j) + V(x_i + x_j)\right) + \sum_{i=1}^{N}\left(\alpha_2^2 V(x_i) + \alpha_3^2 V(2x_i)\right), \tag{1}$$

where

$$V(x) = \wp(x),$$

the Weierstrass function [5]. When $\alpha_1 \neq 0$, to B_n-serie corresponds $\alpha_3 = 0$; to C_n-serie corresponds $\alpha_2 = 0$; to BC_n-series corresponds $\alpha_2 \neq 0$, $\alpha_3 \neq 0$; and finally to D_n-serie corresponds $\alpha_2 = \alpha_3 = 0$. These systems constitute relativistic limits of the generalized Ruijsenaars-Schneider system [1] with the Hamiltonian

$$H = mc^2 \sum_{i=1}^{N} \cosh\left(\frac{\theta_i}{mc}\right) \prod_{j=1, i\neq j}^{N} \left(\sigma\left(\frac{ig}{mc}\right)\right) v(x_i, x_j), \tag{2}$$

where

$$v(x_i, x_j) = \sqrt{a + b\left(\alpha_1^2\left(\wp(x_i - x_j) + \wp(x_i + x_j)\right) + \alpha_2^2 \wp(x_j) + \alpha_3^2(2x_j)\right)}, \tag{3}$$

for

$$a = \left(2\alpha_1^2 + \alpha_2^2 + \alpha_3^2\right)\wp(\eta)\sigma^2(\eta), \tag{4}$$

$$b = -\sigma^2(\eta), \tag{5}$$

$$\eta = \frac{ig}{mc}, \tag{6}$$

$$p = ms \sinh\theta, \tag{7}$$

where θ denotes rapidity.

2. Generalized non-abelian $2D$ Toda model

The generalized non-abelian Toda model chain equation has the form [2, 4]

$$\partial_+((\partial_- g_n)g_n^{-1}) = g_n g_{n-1}^{-1} - g_{n+1}g_n^{-1}, \tag{8}$$

The associated linear problem is then given by [2]

$$\partial_+\psi_n(t_+,t_-) = \psi_{n+1}(t_+,t_-) + v_n(t_+,t_-)\psi_n(t_+,t_-), \tag{9}$$

$$\partial_-\psi_n(t_+,t_-) = c_n(t_+,t_-)\psi_{n-1}(t_+,t_-), \tag{10}$$

where

$$c_n = g_n g_{n-1}^{-1}, \tag{11}$$

$$v_n = (\partial_+ g_n)g_n^{-1}. \tag{12}$$

The constraints (according to [6]) for B_n

$$g_{-n} = g_n^{-1}, \tag{13}$$

$$v_{-n} = -v_n, \tag{14}$$

for C_n series are

$$g_{-n-1} = g_n^{-1}, \tag{15}$$

$$v_{-n} = -v_{n-1}, \tag{16}$$

2.1. Equations of motion

Similarly to Theorems 2.1 and 2.2 of [2] we find the following

Proposition. *The equations of motion for non-abelian 2D Toda system have*

$$\begin{aligned}\partial_t\Psi(x,t) &= \Psi(x+\eta,t) + \sum_{i=1}^{N} a_i(t)b_i^\dagger(t)\left(\alpha_1^2\left(V(x-x_i(t)) + V(x+x_i(t))\right)\right.\\ &+ \left.\alpha_2^2 V(x) + \alpha_3^2 V(2x)\right)\Psi(x,t),\end{aligned} \tag{17}$$

$$\begin{aligned}-\partial_t\Psi^\dagger(x,t) &= \Psi^\dagger(x-\eta,t) + \Psi^\dagger(x,t)\sum_{i=1}^{N} a_i(t)b_i^\dagger(t)\left(\alpha_1^2\left(V(x-x_i(t)) + V(x+x_i(t))\right)\right.\\ &+ \left.\alpha_2^2 V(x) + \alpha_3^2 V(2x)\right),\end{aligned} \tag{18}$$

have N pairs of linearly independent solutions with simple poles

$$\Psi = \sum_{i=1}^{N} s_i(t,k,z)\Phi(x-x_i,z)z^{x/\eta}, \tag{19}$$

$$\Psi^\dagger = \sum_{i=1}^{N} s_i^\dagger(t,k,z)\Phi(-x+x_i(t)-\eta,z)z^{-x/\eta}, \tag{20}$$

$$\Phi(x,z) = \frac{\sigma(z+x+\eta)}{\sigma(z+\eta)\sigma(x)}\left(\frac{\sigma(z-\eta)}{\sigma(z+\eta)}\right)^{x/2\eta}. \tag{21}$$

Note that here

$$\Phi(x,z)=\frac{1}{x}+A+\mathcal{O}(x), \tag{22}$$

where

$$A=\zeta(z+\eta)+\frac{1}{2\eta}\log\frac{\sigma(z-\eta)}{\sigma(z+\eta)}, \tag{23}$$

when $x\longrightarrow 0$.

Proof. By substituting $\Psi(x,t,z,k)$ into equations (17)–(18) and expanding over poles, we obtain (as before) for coefficients near $(x-x_i)^{-2}$:

$$\partial_t x_i s_i=\alpha_1^2 a_i(b_i^\dagger s_i),$$

$$s_{i,\alpha}(t,k,z)=c_i(t,k,z)a_{i,\alpha}(t),$$

$$\partial_t s_i x_i=(b_i^\dagger a_i).$$

For poles $(x-x_i)^{-1}$ we get

$$\begin{aligned}\partial s_i-\alpha_1\left(\sum_{i\neq j,j=1}^{N}a_j(t)b_j^\dagger(t)V(x_i-x_j(t))+\sum_{j=1}^{N}V(x_i+x_j)\right)s_i\\+\sum_{j=1}^{N}a_jb_j^\dagger\left(\alpha_2^2V(x_i)+\alpha_3^2V(2x_i)\right)s_i+(A-\zeta(\eta))\,a_i(t)b_i^\dagger s_i\\-a_i\alpha_1^2\sum_{j\neq i}\left(b_i^\dagger s_j\right)\Phi(x_i-x_j,z)=0,\end{aligned}$$

$$\lambda_i(t)=\frac{\partial_t c_i}{c_i}+(\zeta(\eta)-A)\,\partial_t x_i-\sum_{j\neq i}\alpha_1^2\left(b_i^\dagger\alpha_j\right)\Phi(x_i-x_j,z)\frac{c_j}{c_i}=0.$$

In the matrix form we obtain:

$$(\partial_t+M(t,z))\,C=0,$$

where

$$M_{ij}(t,z)=(-\lambda_j+(\zeta(\eta)-A)\partial_t x_i)\,\delta_{ij}-(1-\delta_{ij})\alpha_1^2b_i^\dagger a_j\Phi(x_i-x_j,z).$$

Canceling of poles $(x-x_i+\eta)^{-1}$ gives

$$-s_ik+\sum_{j\neq i}\alpha_1^2a_ib_i^\dagger s_j\Phi(x_i-x_j-\eta,z)=0,$$

$$(L_- -kI)\,C=0,\qquad C=C(c_i),$$

where

$$L_{-,ij}(t,z)=\alpha_1^2\left(b_i^\dagger a_j\right)\Phi(x_i-x_j-\eta,z).$$

For poles $(x+x_i+\eta)^{-1}$ we obtain:

$$-s_ik+\sum_{j\neq i}\alpha_1^2a_ib_i^\dagger s_j\Phi(-x_i-x_j-\eta,z)=0,$$

$$L_{+,ij}(t,z)=\alpha_1^2\left(b_i^\dagger a_j\right)\Phi(-x_i-x_j-\eta,z.$$

For poles $(x+x_i)^{-1}$ it follows:

$$a_i \sum_{j=1}^{N} \alpha_1^2 \left(b_i^\dagger s_j\right) \Phi(-x_i - x_j, z) k^{x_j/\eta} = 0,$$

i.e.,

$$M_{+,ij}(t,z) = a_i \sum_{j=1}^{N} \alpha_1^2 \left(b_i^\dagger a_j\right) \Phi(-x_i - x_j, z) k^{x_j/\eta},$$
$$M_+ C = 0.$$

For B_n, C_n, BC_n series, near poles x^{-1} for $x \longrightarrow 0$ it follows:

$$\sum_{i \neq j}^{N} a_i \left(b_i^\dagger s_j\right) \Phi(-x_i, z) = 0,$$

i.e.,

$$M_{0,ij}(t,z) = \sum_{j \neq i}^{N} a_i \left(b_i^\dagger a_j\right) \Phi(-x_i, z) = 0,$$
$$M_0 C = 0.$$

For $(x+\eta)^{-1}$:

$$\sum_{i \neq j}^{N} a_i \left(b_i^\dagger s_j\right) \Phi(-\eta - x_i, z) \frac{1}{k} = 0,$$

i.e.,

$$L_{0,ij}(t,z) = \sum_{j \neq i}^{N} a_i \left(b_i^\dagger a_j\right) \Phi(-\eta - x_i, z) \frac{1}{k} = 0,$$
$$L_0 C = 0.$$

Similar results we obtain for equations on $\Psi^\dagger$.

□

2.2. Generalized spin Ruijsenaars models

Proposition. *With*

$$V(x) = \zeta(x) - \zeta(x+\eta),$$

we get for the generalized spin Ruijsenaars models we obtain

$$\begin{aligned}
\partial_t^2 x_i = \alpha_1^2 &\sum_{i \neq j, j=1}^{N} (b_i^\dagger a_j)(b_j^\dagger a_i)(V(x_i - x_j) - V(x_j - x_i)) \\
&+\alpha_1^2 \sum_{j=1}^{N} (b_i^\dagger a_j)(b_j^\dagger a_i) \left(V(x_i + x_j) - V(-(x_i + x_j))\right) \\
&+\alpha_2^2 \left(V(x_i) - V(-x_i)\right) + \alpha_2^3 \left(V(2x_i) - V(-2x_i)\right)\right),
\end{aligned}$$

$$\partial_t a_i = \alpha_1^2 \sum_{j \neq i}^{N} a_j (b_j^\dagger a_i) V(x_i - x_j) + \alpha_1^2 \sum_{j=1}^{N} a_j (b_j^\dagger a_i) V(x_i + x_j)$$

$$+\sum_{i\neq j}^{N}(b_i^\dagger a_j)(b_j^\dagger a_i)\left(\alpha_2^2 V(x_i)+\alpha_3^2 V(2x_i)\right)-\lambda_i a_i,$$

$$\partial_t b_i^\dagger = -\alpha_1^2\sum_{j\neq i}^{N} b_j^\dagger(b_i^\dagger a_j)V(x_j-x_i)-\alpha_1^2\sum_{j=1}^{N} b_j^\dagger(b_i^\dagger a_j)V(-(x_j+x_i))$$

$$+\sum_{j=1}^{N}(b_i^\dagger a_j)(b_j^\dagger a_i)\left(\alpha_2^2 V(-x_i)+\alpha_3^2 V(-2x_i)\right)+\lambda_i^\dagger b_i^\dagger.$$

Proof. Last three equations can be rewritten in the form:

$$\partial_t^2 x_i = \alpha_1^2\sum_{i\neq j,j=1}^{N}(b_i^\dagger a_j)(b_j^\dagger a_i)\left(V(x_i-x_j)+V(x_i+x_j)-V(x_j-x_i)+V(-(x_i+x_j))\right)$$
$$+\sum_{j=1}^{N}(b_i^\dagger a_j)(b_j^\dagger a_i)\left(\alpha_2^2\left(V(x_i)-V(-x_i)\right)+(\alpha_3^2+\alpha_1^2)\left(V(2x_i)-V(-2x_i)\right)\right),$$

$$\partial_t a_i = \alpha_1^2\sum_{j\neq i}^{N} a_j(b_j^\dagger a_i)V(x_i-x_j)+V(x_i+x_j))$$
$$\sum_{j=1}^{N}(b_i^\dagger a_j)(b_j^\dagger a_i)\left(\alpha_2^2 V(x_i)+(\alpha_3^2+\alpha_1^2)V(2x_i)\right)-\lambda_i a_i,$$

$$\partial_t b_i^\dagger = -\alpha_1^2\sum_{j\neq i}^{N} b_j^\dagger(b_i^\dagger a_j)\left(V(x_j-x_i)+V(-(x_j+x_i)\right)$$
$$\sum_{j=1}^{N}(b_i^\dagger a_j)(b_j^\dagger a_i)\left(\alpha_2^2 V(-x_i)+\alpha_3^2 V(-2x_i)\right)+\lambda_i^\dagger b_i^\dagger.$$

□

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