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Relative Entropy and Torsion Coupling

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Abstract. Based on the the geometric realization of entanglement entropy via Ryu-Takayanagi formula, in this work we evaluate the relative entropy for the holographic deformed CFT dual to the torsion gravity coupled to the fermions of nonzero vev in the Einstein-Cartan formulation. We find that the positivity and monotonicity of the relative entropy imposes constraint on the strength of axial-current coupling, fermion mass and equation of state. Our work is the first example to demonstrate the nontrivial constraint on the bulk gravity theory from the quantum information inequalities. Especially, this constraint is beyond the symmetry action principle and should be understood as the unitarity constraint. This talk is based on the work [1] of the authors.

1. Introduction

We evaluate the relative entropy on a ball region near the UV fixed point of a holographic conformal field theory deformed by a fermionic operator of nonzero vacuum expectation value. The positivity of the relative entropy considered here is implied by the expected monotonicity of decrease of quantum entanglement under RG flow. The calculations are done in the perturbative framework of Einstein-Cartan gravity in four-dimensional asymptotic anti-de Sitter space with a postulated standard bilinear coupling between axial fermion current and torsion, i.e.,

$$\mathcal{L}_{\psi K} = \frac{\eta_t}{4!}\sqrt{-g}\,\epsilon_{\mu\nu\rho\sigma}\,\bar{\psi}\gamma^{\mu}\gamma_5\psi\,K^{\nu\rho\sigma} \tag{1}$$

where $K^{\nu\rho\sigma}$ is the contorsion tensor. If $\eta_t = 1$, $\mathcal{L}_{\psi K}$ is the canonical coupling of the fermion to torsion as conventionally chosen in [2]. Holographically, this is dual to deforming the holographic CFT by such a coupling term

$$\delta\lambda \int d^d x\; \mathcal{O}_{\Delta}(x) \tag{2}$$

and with

$$\langle \int d^d x\; \mathcal{O}_{\Delta}(x)\rangle \neq 0 \tag{3}$$

where $\mathcal{O}_{\Delta}$ is some single-trace fermion operator of conformal dimension Δ, and $\delta\lambda$ is a spinor characterizes the amount of deformation.

By requiring positivity of relative entropy, our result yields a constraint on axial current-torsion coupling, fermion mass and equation of state. In the following we sketch our results.

2. Relative Entropy and Holographic Consideration

The relative entropy is a "distance" measure on the (quantum) state space. For two state ρ and σ, the relative entropy is defined as

$$S(\rho||\sigma) := \mathrm{tr}(\rho \log \rho) - \mathrm{tr}(\rho \log \sigma) \ . \tag{4}$$

As a "distance" measure, the relative entropy is non-negative, i.e.,

$$S(\rho||\sigma) \geq 0 \tag{5}$$

in which the equality holds if and only if $\rho = \sigma$.

For a quantum state one can express the reduced density matrix σ_A on the region A in terms of the modular Hamiltonian H_A as follows:

$$\sigma_A = \frac{e^{-H_A}}{\mathrm{tr}\, e^{-H_A}} \ , \tag{6}$$

then the relative entropy can be rewritten as

$$S(\rho_A||\sigma_A) = \Delta\langle H_A\rangle - \Delta S_A \tag{7}$$

where

$$\Delta\langle H_A\rangle \ := \ -\mathrm{tr}(\rho_A \log \sigma_A) + \mathrm{tr}(\sigma_A \log \sigma_A) \ , \tag{8}$$

$$\Delta S_A \ := \ -\mathrm{tr}(\rho_A \log \rho_A) + \mathrm{tr}(\sigma_A \log \sigma_A) \ . \tag{9}$$

Note that the entanglement entropy for state σ on the region A is defined by

$$S_A(\sigma) := -\mathrm{tr}(\sigma_A \log \sigma_A) \tag{10}$$

so that ΔS_A is the difference of entanglement entropy of the region A between states ρ and σ. Via (7) the positivity of relative entropy yields

$$\Delta\langle H_A\rangle \geq \Delta S_A \ . \tag{11}$$

The modular Hamiltonian H_A is in general nonlocal and unknown. However, for CFT$_d$ with the interested region A to be a disk of radius R_A, it has a closed form as follows [3]:

$$H_A = 2\pi \int_{|x|<R_A} d^{d-1}x \ \frac{R_A^2 - r^2}{2R_A} \ T_{tt}(\vec{x}) \tag{12}$$

where T_{tt} is the (holographic) energy density operator of CFT.

To extract some constraints from the relative entropy, we are interested in the case where σ is the CFT vacuum state, and ρ is a perturbative state (by either excitation or deformation) away from σ, i.e.,

$$\rho = \sigma + \delta\rho \ , \qquad |\delta\rho|/|\rho| << 1 \ . \tag{13}$$

Since the relative entropy takes its extremum at $\rho = \sigma$, its first order variation vanishes, which then yields the first law of entanglement thermodynamics [4], i.e.,

$$\Delta\langle H_A\rangle|_{\mathcal{O}(\delta\rho)} = \Delta S_A|_{\mathcal{O}(\delta\rho)} \ . \tag{14}$$

On the other hand, the positivity of the second order variation will impose some unitarity bound on the deformed states of CFT.

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In the context of AdS/CFT correspondence, a dual state is characterized by an asymptotic AdS bulk metric, i.e., in the Poincare coordinates,

$$ds^2 = \frac{\ell^2}{z^2}\Big(G(z)dz^2 + H_{\mu\nu}(z, x^\mu)dx^\mu dx^\nu\Big) \tag{15}$$

where ℓ is the AdS radius. The CFT vacuum state σ corresponds to $G(z) = 1$ and $H_{\mu\nu} = \eta_{\mu\nu}$. The perturbative state ρ will be given by some slightly deviated $H_{\mu\nu}$ as well as $G(z)$.

We can then obtain the modular Hamiltonian H_A on the region A from holographic $T_{\mu\nu}$ for the corresponding state by the relation (12) if A is a ball. By evaluating H_A for both ρ and σ states and subtracting them, we can then obtain $\Delta\langle H_A\rangle$. We still need to calculate ΔS_A holographically in order to obtain the relative entropy by (7). The entanglement entropy can be obtained holographically by the Ryu-Takayanagi formula [5]:

$$S_A = \frac{\mathrm{Area}(\gamma_A)}{4G_N} \tag{16}$$

where γ_A is the co-dimensional two extremal surface ending on the entangling surface ∂_A at the AdS boundary $z = 0$. For static metric, γ_A is simply the minimal surface on a fixed time slice. We can then evaluate the γ_A and thus S_A for the bulk metrics corresponding to states ρ and σ, and then subtract them to obtain ΔS_A.

3. Summary of Our Results

In this work we consider the deformed holographic CFT with fermionic sources in AdS_4 space in the framework of Einstein-Cartan gravity with a postulated coupling constant for the bilinear interaction between axial current and torsion. Note that this coupling cannot be fixed by the dynamical symmetry of holographic CFT. Instead we find that the positivity of relative entropy disfavors such a coupling.

By turning on both normalizable and non-normalizable fermionic zero modes, the bulk geometry will be backreacted by the torsion at the second order of gravitational coupling. We will solve the backreacted metric in the expansion of Newton constant, denoted by K, up to K^2 order, i.e.,

$$\mathbf{g} = \mathbf{g}_0 + \mathrm{K}\,\mathbf{g}_1 + \mathrm{K}^2\,\mathbf{g}_2 \tag{17}$$

where $\mathbf{g}_0$ is the pure AdS_4 metric. Note that $\mathbf{g}_1$ and $\mathbf{g}_2$ will encode the information about the fermionic sources.

Moreover, a bulk fermion of mass m considered here is dual to a CFT operator of conformal dimension Δ given by

$$\Delta = \frac{d}{2} + |m|\ . \tag{18}$$

Unlike the case for the scalar field/operator, from (18) we see no analogue of BF bound on m for preventing from instability.

In this paper, we will consider the backreaction due to the non-vanishing stress tensor of the bulk fermionic zero modes. This is dual to the holographic deformed CFT given by (2) and (3). Then, we will evaluate the relative entropy up to K^2 (or δl^2) order by using the backreacted metric.

As we will see, the torsion will affect $\mathbf{g}_2$ but not $\mathbf{g}_1$. Thus, at the K order, everything should go as the usual Einstein gravity so that the relative entropy evaluated holographically will satisfy the first law (14), i.e.,

$$S(\rho||\sigma)|_{\mathrm{K}} = \Delta\langle H_A\rangle|_{\mathrm{K}} - \Delta S_A|_{\mathrm{K}} = 0\ , \tag{19}$$

as expected.

For the evaluation of the second order relative entropy, we first need to evaluate $\Delta\langle H_A\rangle|_{\mathrm{K}^2}$ by plugging $\langle T_{tt}\rangle|_{\mathrm{K}^2}$ into (12). However, the holographic evaluation of $\langle T_{tt}\rangle|_{\mathrm{K}^2}$ via $\mathbf{g}_2$ yields zero. Thus, we have $\Delta\langle H_A\rangle|_{\mathrm{K}^2} = 0$. Combined this result with the one of (19), we obtain the relative entropy (7) up to K^2 order, that is

$$S(\rho||\sigma) = -\Delta S_A|_{\mathrm{K}^2} \geq 0\ . \tag{20}$$

To evaluate $\Delta S_A|_{\mathrm{K}^2}$, we need to first evaluate the the minimal surface γ_A with respect to the metric $\mathbf{g}_0 + \mathrm{K}\mathbf{g}_1$ up to K^2 order, and we denote it by $\gamma_A^{(1)}$. Then, we can obtain $\Delta S_A|_{\mathrm{K}^2}$ by

$$\Delta S_A|_{\mathrm{K}^2} := \frac{\mathrm{Area}(\gamma_A^{(1)})|_{\mathbf{g}}}{4G_N}|_{\mathrm{K}^2} \tag{21}$$

where $\mathrm{Area}(\gamma_A^{(1)})|_{\mathbf{g}}$ means evaluating the area of surface $\gamma_A^{(1)}$ with respect to the metric (17).

In this work we find that the constraint (20) yields a constraint on m:

$$m^2\ell^2 \geq \frac{2\eta_t^2}{\mu_0^2} \tag{22}$$

where η_t is the postulated coupling constant for the bilinear interaction between bulk axial current and torsion, and μ_0 characterizes the equation of state for the dual deformed fermion state by

$$P = \frac{\mu_0 - 2}{2\mu_0}\ \varepsilon\ . \tag{23}$$

Here P is the pressure and ε is the energy density of the deformed fermion state. Note that the monotonicity condition $\frac{\partial S(\rho||\sigma)}{\partial R_A} \geq 0$ yields the same condition (22).

We see that the positivity of the relative entropy imposes constraint on the bilinear coupling η_t in (1) as well as the fermion mass m and equation of state μ_0. If there were no such coupling, i.e, $\eta_t = 0$, then the relative entropy is positive for all m, which is consistent with the fact there is no BF-bound for fermion's mass and seems more natural. Otherwise, the constraint is on all three parameters in a nontrivial way. It is interesting to see how this constraint can be understood as some energy condition in the bulk Einstein-Cartan gravity.

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