Robin Gravity

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Abstract. We write down a Robin boundary term for general relativity. The construction relies on the Neumann result of arXiv:1605.01603 in an essential way. This is unlike in mechanics and (polynomial) field theory, where two formulations of the Robin problem exist: one with Dirichlet as the natural limiting case, and another with Neumann.

1. Introduction

The Dirichlet boundary term for general relativity was found by York and Gibbons-Hawking[1] long ago, but the Neumann term was only written down recently [2]. See [3, 4] for follow-ups. In this contribution, we will further develop the result of [2] to construct a well-defined Robin variational problem and construct the general Robin boundary term for general relativity. The Neumann term turns out to be essential for the Robin construction in gravity in a way that it is not, in mechanics and field theory. To clarify this, we outline the various constructions systematically. We also briefly comment on aspects of such a term in asymptotically flat and asymptotically AdS spacetimes.

This contribution is a small application of the results in [2, 3, 4]. But the existence of a Robin term for general relativity does not seem to have been appreciated in the literature, so we hope it will be of some use to someone somewhere sometime.

2. Robin Mechanics

Let us start by looking at boundary terms in the simplest setting: particle mechanics. Consider the usual Newton action

$$S_D^p[q] = \int_T dt\, L(q, \dot{q}) = \int_T dt\, \left(\frac{1}{2}\dot{q}^2 - V(q) \right). \tag{1}$$

The superscript p indicates the action is that of a particle, and the subscript D denotes that it leads to a well defined variational problem with *Dirichlet* boundary condition (in the time direction). To restate the well-known, the variation of the action gives

$$\delta S_D^p = -\int_T dt\, \left(\ddot{q} + V'(q) \right)\delta q + (\dot{q}\delta q)\Big|_T. \tag{2}$$

If one sets $q = any\ fixed\ quantity$ at the endpoint[1] T, the variational problem becomes well posed, and since we are setting $\delta q|_T = 0$ we call it a Dirichlet problem.

Note that setting $\dot{q}|_T = 0$ in δS_D^p is another way to define a valid variational problem, while *not* demanding[2] that $\delta q = 0$. We will call this the *Special Neumann* boundary condition.

We would like to find a variational principle where holding $\delta \dot{q} = 0$ is well-defined. This is the natural *General Neumann* boundary condition, and to accomplish this we add a boundary term to the action:

$$S_N^p \;=\; S_D^p - (q\ \dot{q})\Big|_T, \tag{3}$$

$$\Rightarrow \delta S_N^p \;=\; -\int_T dt\ \Big(\ddot{q} + V'(q)\Big)\delta q - (q\delta\dot{q})\Big|_T. \tag{4}$$

We could restate it in terms of conjugate quantities at the boundary, which leads to a more useful notation later, as

$$(q\ \dot{q})\Big|_T \equiv \pi_T\, q_T,\ \text{where}\ \pi_T \equiv \frac{\delta S_D^p}{\delta q_T}. \tag{5}$$

Note that General Neumann boundary conditions basically mean fixing $\dot{q} = any\ fixed\ value$, while Special Neumann boundary condition allows only the possibility $\dot{q} = 0$.

Now, let us consider another boundary term that we could add to S_D^p, namely $S_1^p = \frac{\xi}{2}q^2|_T$. This generalizes the Special Neumann boundary condition and leads to what we will call the *Special Robin* boundary condition. Upon varying $S_D^p + S_1^p$ we get

$$\delta(S_D^p + S_1^p) = -\int_T dt\ \Big(\ddot{q} + V'(q)\Big)\delta q + \Big(\dot{q} + \xi\ q\Big)\delta q\Big|_T. \tag{6}$$

If we set $\delta q = 0$, this is still the Dirichlet variational problem. But we can also set $(\dot{q}+\xi q)|_T = 0$, which is the Special Robin boundary condition: holding a linear combination of the position and velocity fixed at the boundary. When $\xi = 0$, this reduces to the Special Neumann boundary condition.

What is the Robin analogue of the General Neumann boundary condition? Lets consider adding one more piece to our General Neumann action[3]:

$$S_R^p = \int_T dt\left(\frac{1}{2}\dot{q}^2 - V(q)\right) - \Big(\dot{q}\,q + \frac{\xi}{2}q^2\Big)\Big|_T, \tag{7}$$

which upon varying gives

$$\delta S_R^p = -\int_T dt\ \Big(\ddot{q} + V'(q)\Big)\delta q - q\,\delta\Big(\dot{q} + \xi q\Big)\Big|_T. \tag{8}$$

The variational problem is well defined by setting $\dot{q} + \xi q = any\ fixed\ value$. This is the *General Robin* boundary condition. Again, we could phrase the whole thing as

$$\delta S_R^p = \text{eoms} - q_T\,\delta(\pi_T + \xi q_T) = \text{eoms} - q_T\,\delta\left(\frac{\delta S_D^p}{\delta q_T} + \xi q_T\right). \tag{9}$$

---

[1] We keep track of only one boundary, as it is suffices to make our point.

[2] Demanding both $\delta q = 0 = \dot{q}$ fixes both the function and its derivative at the boundary, constraining dynamics uniquely. This is not what we want from a theory: it should allow dynamics, not uniquely fix it.

[3] It is possible to set up a General Robin boundary problem for particle mechanics, by starting with the Dirichlet action and never going through the Neumann action. However, this approach does not work for gravity and we find that the Neumann action is crucial for the construction of the Robin action for general relativity. We discuss these matters in Appendix A.

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The Dirichlet problem can be understood as a variational problem with the position of the particle held arbitrary and fixed at T, and the General Neumann problem to be a variational problem with the momentum at T held arbitrary and fixed. The General Robin boundary condition is analogously to be thought of as holding some linear combination of the position and momentum held arbitrary and fixed at T. The Dirichlet [1] and General Neumann problem [2] for general relativity are solved, here we would like to fill the gap and formulate the General Robin problem for gravity. There is a bit of a subtlety in this compared to the particle mechanics case (see footnote 3).

But before getting to gravity, we consider the field theory case which is essentially just a fancy rewriting of the particle mechanics case.

3. Robin Field Theory
We will start with the action for a scalar field living in a D-dimensional manifold $(\mathcal{M}, g)$, which again is automatically a Dirichlet action, where we hold the field to be at some fixed value at the boundary

$$S_D[\phi] = \int_{\mathcal{M}} d^D x \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right), \tag{10}$$

$$\Rightarrow \delta S_D[\phi] = \int_{\mathcal{M}} d^D x \left(\partial_\mu(\sqrt{-g} g^{\mu\nu} \partial_\nu \phi) - \frac{\partial V(\phi)}{\partial \phi} \right) \delta \phi - \int_{\partial \mathcal{M}} d^{D-1} x \sqrt{|\gamma|} n_\mu g^{\mu\nu} \partial_\nu \phi \, \delta \phi \tag{11}$$

where γ is the metric on the boundary $\partial \mathcal{M}$ of $\mathcal{M}$, and n_μ is the outward drawn normal to the boundary. The standard procedure, as mentioned earlier is to set the Dirichlet boundary condition $\delta \phi = 0$, which leads to a well defined variational problem. The Special Neumann case is obtained from the same action while setting the directional derivative $\partial_n \phi \equiv n_\mu g^{\mu\nu} \partial_\nu \phi = 0$ instead. As before, if we work with

$$S_D[\phi] + \int_{\partial \mathcal{M}} d^{D-1} x \sqrt{|\gamma|} \, \frac{\xi}{2} \phi^2 \tag{12}$$

it leads to the Special Robin variational problem with $\partial_n \phi + \xi \phi = 0$ at the boundary. To get the General Neumann action, we take by direct analogy

$$S_N[\partial_n \phi] = S_D[\phi] + \int_{\partial \mathcal{M}} d^{D-1} x \sqrt{|\gamma|} (n_\mu g^{\mu\nu} \partial_\nu \phi) \phi \tag{13}$$

$$\Rightarrow \delta S_N[\partial_n \phi] = \text{eom} + \int_{\partial \mathcal{M}} d^{D-1} x \sqrt{|\gamma|} \, \delta(\partial_n \phi) \phi. \tag{14}$$

The variational problem here is well defined by holding $\delta(\partial_n \phi) = 0$. The scalar field theory can also be well posed as a General Robin boundary problem:

$$S_R[\phi] = S_D[\phi] + \int_{\partial \mathcal{M}} d^{D-1} x \sqrt{|\gamma|} \left(\phi \, \partial_n \phi + \frac{\xi}{2} \phi^2 \right) = S_N[\phi] + \int_{\partial \mathcal{M}} d^{D-1} x \sqrt{|\gamma|} \, \frac{\xi}{2} \phi^2, \tag{15}$$

which will lead to holding $\partial_n \phi + \xi \phi = any\ fixed\ value$ at $\partial \mathcal{M}$. We worked with the scalar for simplicity, but this generalizes trivially to the gauge field as well.

4. Robin Gravity
We can now proceed to look for a boundary term that gives a consistent Robin boundary problem for gravity. Let us, as usual, start with the Einstein-Hilbert action on a D-dimensional manifold $(\mathcal{M}, g)$ along with the Gibbons-Hawking York boundary term, which leads to Dirichlet gravity

$$S_D = S_{EH} + S_{GHY} = \frac{1}{2\kappa} \int_{\mathcal{M}} d^D x \sqrt{-g}(R - 2\Lambda) + \frac{1}{\kappa} \int_{\partial \mathcal{M}} d^{D-1} y \sqrt{|\gamma|} \, \epsilon \Theta, \tag{16}$$

where $\kappa = 8\pi G_N$, R is the Ricci scalar and Λ is a cosmological constant. Also, $\gamma_{ij} = g_{\mu\nu} e_i^\mu e_j^\nu$ is the induced metric on the boundary $\partial\mathcal{M}$ and $e_i^\mu = \frac{\partial x^\mu}{\partial y^i}$ projects the bulk coordinates x^μ to the boundary coordinates y^i. The extrinsic curvature of the boundary is given by

$$\Theta_{ij} = \frac{1}{2}(\nabla_\mu n_\nu + \nabla_\nu n_\mu) e_i^\mu e_j^\nu, \tag{17}$$

where n_μ is the outward drawn unit normal to the boundary, and $\epsilon = \pm 1$ distinguishes the boundary between time-like and space-like boundaries respectively.

The variation of Dirichlet action yields

$$\begin{aligned} \delta S_D &= \delta S_{EH} + \delta S_{GHY} \\ &= \frac{1}{2\kappa} \int_\mathcal{M} d^D x \sqrt{-g}\, (G_{\mu\nu} + \Lambda g_{\mu\nu})\, \delta g^{\mu\nu} - \frac{1}{2\kappa} \int_{\partial\mathcal{M}} d^{D-1} y \sqrt{|\gamma|} \epsilon \left(\Theta^{ij} - \Theta \gamma^{ij} \right) \delta\gamma_{ij}, \end{aligned} \tag{18}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor. The variational problem is well defined with the boundary metric held fixed, and we can think of $S_D = S_D[\gamma_{ij}]$ as a functional of the boundary metric.

We can define a canonical conjugate of the boundary metric as

$$\pi^{ij} \equiv \frac{\delta S_D}{\delta\gamma_{ij}} = -\frac{1}{2\kappa} \sqrt{|\gamma|} \epsilon \left(\Theta^{ij} - \Theta \gamma^{ij} \right), \tag{19}$$

using which we can rewrite the variation of S_D in a simpler form

$$\delta S_D = \frac{1}{2\kappa} \int_\mathcal{M} d^D x \sqrt{-g}\, (G_{\mu\nu} + \Lambda g_{\mu\nu})\, \delta g^{\mu\nu} + \int_{\partial\mathcal{M}} d^{D-1} y\, \pi^{ij}\, \delta\gamma_{ij}. \tag{20}$$

We also note that holding $\pi^{ij} = 0$ here leads to the Special Neumann boundary condition for gravity. This is sometimes described as the Neumann problem for gravity in the literature, even though it is a special case of the general situation.

As was discussed in [2, 3, 4], an action which is well defined in terms of General Neumann boundary condition can be defined as

$$\begin{aligned} S_N &= S_{EH} + S_{GHY} - \int_{\partial\mathcal{M}} d^{D-1} y\, \pi^{ij}\, \gamma_{ij} \tag{21} \\ &= \frac{1}{2\kappa} \int_\mathcal{M} d^D x \sqrt{-g}(R - 2\Lambda) + \frac{4 - D}{2\kappa} \int_{\partial\mathcal{M}} d^{D-1} y \sqrt{|\gamma|} \epsilon \Theta, \tag{22} \end{aligned}$$

the variation of which is given by

$$\delta S_N = \frac{1}{2\kappa} \int_\mathcal{M} d^D x \sqrt{-g}\, (G_{\mu\nu} + \Lambda g_{\mu\nu})\, \delta g^{\mu\nu} + \int_{\partial\mathcal{M}} d^{D-1} y\, \gamma_{ij}\, \delta\pi^{ij}. \tag{23}$$

Here, instead of holding the boundary metric fixed, the quantity π^{ij} is held fixed, letting the boundary metric fluctuate. The quantity π^{ij} is termed boundary stress tensor density (also, sometimes as quasi-local stress tensor density[5]). The Neumann boundary condition can be thought of as looking at solutions holding the boundary stress tensor density fixed, i.e. $S_N = S_N[\pi^{ij}]$.

Now we turn to Special Robin. Adding a boundary term $S_b = 2\zeta \int d^{D-1} y \sqrt{|\gamma|}$ to the Dirichlet action, it is straightforward to again check that we will have a variational problem well defined under the Special Robin boundary condition, $\pi^{ij} + \zeta \sqrt{|\gamma|}\gamma^{ij} = 0$.

In order to have the action be a well defined variational problem under General Robin boundary condition, we need to add a boundary term which will ensure that $\pi^{ij} + \xi\sqrt{|\gamma|}\gamma^{ij}$ held arbitrary and fixed[4] leads to a consistent variational problem. To get such an action, we go through the Neumann action like we did in the mechanics and field theory cases. Note that unlike in those cases, in gravity we cannot get to Robin from Dirichlet bypassing Neumann[5]. In other words, going through Neumann is not an option but a necessity in the case of gravity.

In any event, the result is

$$
\begin{aligned}
S_R &= \frac{1}{2\kappa}\int_{\mathcal{M}} d^D x\,\sqrt{-g}(R - 2\Lambda) \\
&\quad + \frac{4-D}{2\kappa}\int_{\partial\mathcal{M}} d^{D-1}y\sqrt{|\gamma|}\epsilon\Theta - \xi(D-3)\int_{\partial\mathcal{M}} d^{D-1}y\sqrt{|\gamma|}.
\end{aligned}
\tag{24}
$$

Varying the action, and using the key relation

$$
(D-3)\sqrt{|\gamma|}\gamma^{ij}\delta\gamma_{ij} = 2\,\delta(\sqrt{|\gamma|}\gamma^{ij})\,\gamma_{ij},
\tag{25}
$$

we can show that

$$
\delta S_R = \frac{1}{2\kappa}\int_{\mathcal{M}} d^D x\,\sqrt{-g}\,(G_{\mu\nu} + \Lambda g_{\mu\nu})\,\delta g^{\mu\nu} - \int_{\partial\mathcal{M}} d^{D-1}y\,\delta(\pi^{ij} + \xi\sqrt{|\gamma|}\gamma^{ij})\,\gamma_{ij}.
\tag{26}
$$

The action (24) is what we call the Robin action for gravity.

5. Comments
Hamiltonian Formulation
We will now write down the Robin Gravity action in the Hamiltonian formulation. Using the fact that the action in (24) is the same as that of Neumann gravity, except for an additional boundary cosmological constant term, we can directly write down the action in terms of canonical variables for Robin gravity [3]:

$$
\begin{aligned}
S_R &= \int_{\mathcal{M}} d^D x\left(p^{ab}\dot{h}_{ab} - NH - N_a H^a\right) \\
&\quad + \int_{\mathcal{B}} d^{D-1}y\sqrt{\sigma}\left(N\left(\frac{\varepsilon}{2} - \xi(d-3)\right) - N^a j_a + \frac{N}{2}s^{ab}\sigma_{ab}\right).
\end{aligned}
\tag{27}
$$

We will not elaborate on the (completely standard) notations here, they can be found in, eg. [3].
Asymptotically Flat Space-times
If one goes about naively computing the classical action for any asymptotically flat space-time(AFS), its bound to run into divergences. The usual procedure to deal with in AFS is to do a background subtraction, which involves holding the induced metric at the boundary the same for the background and the datum. For (24), this means that the boundary cosmological constant term drops off and we end up with the same result as what one would get from a pure Neumann boundary term, see [3]. This means that various discussions there on thermodynamics, horizons, etc [6] also immediately apply to the background subtracted case here.

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[4] The explicit presence of $\sqrt{|\gamma|}$ is not of much worry, as one can see, π^{ij} is also defined implicitly with the same factor.

[5] See discussion in the Appendix for some elaboration on this.

AdS

We will now look at asymptotically AdS_{d+1} spaces[6]. We follow the notations of [4].

The renormalized Neumann action is given by

$$S_N^{ren} = S_D^{ren} - \int_{\partial\mathcal{M}} d^dx\, \pi_{ren}^{ij}\gamma_{ij}, \tag{28}$$

where π_{ren}^{ij} is the renormaized boundary stress tensor density.

The boundary stress tensor is related to π^{ij} as[7]

$$T^{ij} = -\frac{2}{\sqrt{-\gamma}}\pi^{ij} = \frac{1}{\kappa}(\Theta^{ij} - \Theta\,\gamma^{ij}), \tag{29}$$

and the renormalized stress tensor T_{ren}^{ij} is obtained from π_{ren}^{ij} in the same way. The variation of renormalized Neumann action gives

$$
\begin{aligned}
\delta S_N^{ren} &= \text{ eq. of motion } - \int_{\partial\mathcal{M}} d^dx\, \gamma_{ij}\delta\pi_{ren}^{ij} \\
&= \text{ eq. of motion } + \frac{1}{2}\int_{\partial\mathcal{M}} d^dx\, g_{0\,ij}\delta(\sqrt{-g_0}\mathcal{T}^{ij}),
\end{aligned} \tag{30}
$$

where $\mathcal{T}^{ij}$ is the true renormalized stress tensor (of the boundary CFT) and is given by[7]

$$\mathcal{T}_{ij} = \lim_{\epsilon\to 0}\left(\frac{1}{\epsilon^{\frac{d}{2}-1}}T_{ij}^{ren}[\gamma]\right) = \lim_{\epsilon\to 0}\left(-\frac{2}{\sqrt{g(x,\epsilon)}}\frac{\delta S_D^{ren}}{\delta g^{ij}}\right) = -\frac{2}{\sqrt{-g_0}}\frac{\delta S_D^{ren}}{\delta g_0^{ij}}. \tag{31}$$

We can write down the Robin gravity action specific to AdS as

$$S_R^{ren} = S_D^{ren} - \int_{\partial\mathcal{M}} d^dx\, \pi_{ren}^{ij}\gamma_{ij} + \frac{(d-2)}{2}\xi\int_{\partial\mathcal{M}} d^dx\sqrt{-g_0}, \tag{32}$$

which upon variation gives

$$\delta S_R = \text{eq. of motion} + \frac{1}{2}\int_{\partial\mathcal{M}} d^dx\, g_{0\,ij}\,\delta\left(\sqrt{-g_0}(\mathcal{T}^{ij} + \xi g_0^{ij})\right), \tag{33}$$

with the variational principle well defined by holding $\sqrt{-g_0}(\mathcal{T}^{ij} + \xi g_0^{ij}) = any\ fixed\ quantity$.

The essential difference between flat space and AdS is that here the variational principle is best formulated in terms quantities that are intrinsic to the field theory: in other words, in terms of a combination of the g_0 and the g_d in the Fefferman-Graham expansion (see [4, 7]) instead of induced metric γ. Note that $\mathcal{T}^{ij}$ is determined in terms of them [7].

---

[6] We set $D = d + 1$ for convenience.
[7] The boundary we are looking at is time-like $\epsilon = +1$, in the entire AdS discussion.

Appendix A. Another Path to Robin?

In the particle mechanics and field theory cases, there exists a direct path from the Dirichlet action to the General Robin action. Let us start with the particle mechanics problem. To the Dirichlet action in (1), add a boundary term $S_B^p = \frac{\xi}{2}\dot{q}^2\big|^T$, the variation of the sum of two gives

$$\delta(S_D^p + S_B^p) = -\int_T dt \left(\ddot{q}^2 + V'(q)\right)\delta q + \dot{q}\,\delta\left(\xi\dot{q} + q\right)\Big|_T. \tag{A.1}$$

This is clearly a well-defined General Robin variational principle. This sort of thing extends trivially to the field theory case as well. Simply consider the following addition to the Dirichlet field theory action:

$$S_R' = \int_{\mathcal{M}} d^D x\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right) - \frac{\xi}{2}\int_{\partial\mathcal{M}} d^{D-1}x\sqrt{|\gamma|}(\partial_n\phi)^2, \tag{A.2}$$

$$\Rightarrow \delta S_R' = \int_{\mathcal{M}} d^D x\left(\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) - \frac{\partial V(\phi)}{\partial\phi}\right)\delta\phi + $$
$$- \int_{\partial\mathcal{M}} d^{D-1}x\sqrt{|\gamma|}\partial_n\phi\,\delta\left(\phi + \xi\partial_n\phi\right) \tag{A.3}$$

For the case of gravity, one might think that an analogous boundary piece can be added to the Dirichlet (Gibbons-Hawking-York) action to produce the General Robin action. This however does not seem to work: for gravity, we find that going through the Neumann action seems to be essential to obtain the Robin action. We describe why this is so, below.

There are two possible terms that could be constructed out of π_{ij}'s that are quadratic, namely, $(\pi^{ij}\gamma_{ij})^2$ and $\pi^{ij}\gamma_{jk}\pi^{kl}\gamma_{li}$. Also, one has to remember that π^{ij} internally contains a $\sqrt{|\gamma|}$ factor, so it would be more advisable to write boundary terms using T^{ij}, the boundary stress tensor[8]. Let us look at the variation of the first candidate, modulo the constants

$$\delta S_1 = \int_{\partial\mathcal{M}} d^{D-1}x\,\delta\big[\sqrt{|\gamma|}\,[\mathrm{Tr}(T)]^2\big]$$
$$= \int_{\partial\mathcal{M}} d^{D-1}x\sqrt{|\gamma|}\left(\frac{1}{2}[\mathrm{Tr}(T)]^2\gamma^{ij}\delta\gamma_{ij} + 2\,\mathrm{Tr}(T)\big(T^{ij}\delta\gamma_{ij} + \gamma_{ij}\delta T^{ij}\big)\right). \tag{A.4}$$

This can be written in terms of π^{ij}'s as

$$\delta S_1 = 4\int_{\partial\mathcal{M}} d^{D-1}x\,\delta\left[\frac{1}{\sqrt{|\gamma|}}(\pi^{ij}\gamma_{ij})^2\right]$$
$$= 4\int_{\partial\mathcal{M}} d^{D-1}x\,\frac{1}{\sqrt{|\gamma|}}\left(-\frac{1}{2}(\pi^{ij}\gamma_{ij})^2\gamma^{kl}\delta\gamma_{kl} + 2(\pi^{ij}\gamma_{ij})\pi^{kl}\delta\gamma_{kl} + 2(\pi^{ij}\gamma_{ij})\gamma_{kl}\delta\pi^{kl}\right). \tag{A.5}$$

The variation of the second candidate term gives,

$$\delta S_2 = \int_{\partial\mathcal{M}} d^{D-1}x\,\delta\big[\sqrt{|\gamma|}\,\mathrm{Tr}(T^2)\big]$$
$$= \int_{\partial\mathcal{M}} d^{D-1}x\sqrt{|\gamma|}\left(\frac{1}{2}\mathrm{Tr}(T^2)\gamma^{ij}\delta\gamma_{ij} + 2\,T^{ij}\delta T_{ij} + 2\,T^{ij}T_j{}^k\delta\gamma_{ik}\right). \tag{A.6}$$

---

[8] In the following we use the defintions $[\mathrm{Tr}(T)]^2 = (T^{ij}\gamma_{ij})^2$ and $\mathrm{Tr}(T^2) = T^{ij}\gamma_{jk}T^{kl}\gamma_{li}$.

This can be written in terms of π^{ij}'s as

$$
\begin{aligned}
\delta S_2 &= 4 \int_{\partial \mathcal{M}} d^{D-1}x \; \delta \left[\frac{1}{\sqrt{|\gamma|}} \left(\pi^{ij} \gamma_{jk} \pi^{kl} \gamma_{li} \right) \right] \\
&= 4 \int_{\partial \mathcal{M}} d^{D-1}x \; \frac{1}{\sqrt{|\gamma|}} \Big(-\frac{1}{2} (\pi^{ij} \gamma_{jm} \pi^{mn} \gamma_{ni}) \gamma^{kl} \delta \gamma_{kl} \\
&\qquad\qquad\qquad\qquad\qquad + 2\pi^{ij} \gamma_{jk} \pi^{kl} \delta \gamma_{li} + 2\pi^{ij} \gamma_{jk} \gamma_{li} \delta \pi^{kl} \Big).
\end{aligned}
\tag{A.7}
$$

To allow the most general possibility, let us consider adding a combination of these two candidate terms with arbitrary coefficients to the Dirichlet action:

$$
S'_R = S_D + \xi \int_{\partial \mathcal{M}} d^{D-1}x \; \frac{1}{\sqrt{|\gamma|}} \left(\pi^{ij} \gamma_{ij} \right)^2 + \zeta \int_{\partial \mathcal{M}} d^{D-1}x \; \frac{1}{\sqrt{|\gamma|}} \; \pi^{ij} \gamma_{jk} \pi^{kl} \gamma_{li}.
\tag{A.8}
$$

Upon variation this yields

$$
\begin{aligned}
\delta S'_R &= \text{eom} + \int_{\partial \mathcal{M}} d^{D-1}x \; \pi^{ij} \Bigg\{ \left[\delta_i^k \delta_j^l + \frac{\xi}{\sqrt{|\gamma|}} \left(-\frac{1}{2} \pi^{mn} \gamma_{mn} \gamma_{ij} \gamma^{kl} + 2\pi^{mn} \gamma_{mn} \delta_i^k \delta_j^l \right) \right. \\
&\qquad\qquad\qquad\qquad \left. + \frac{\zeta}{\sqrt{|\gamma|}} \left(-\frac{1}{2} \pi^{mn} \gamma_{im} \gamma_{jn} \gamma^{kl} + 2\pi^{nl} \gamma_{jn} \delta_i^k \right) \right] \delta \gamma_{kl} \\
&\qquad\qquad\qquad\qquad + \frac{2}{\sqrt{|\gamma|}} \left[\xi \gamma_{ij} \gamma_{kl} + \zeta \gamma_{ik} \gamma_{jl} \right] \delta \pi^{kl} \Bigg\}
\end{aligned}
\tag{A.9}
$$

For this to reduce to a General Robin variation, we need the coefficient of $\delta \pi^{kl}$ to be some number times the coefficient of $\delta \gamma_{kl}$. This is clearly impossible for any choice of ζ and ξ.

The essential difference between mechanics/field theory and gravity is that here, the $\sqrt{|\gamma|}$ term shows up, which is essentially non-polynomial. This makes the Neumann term an essential intermediate step in our path to Robin: there does not seem to be direct path to it from the Dirichlet (Gibbons-Hawking) boundary term.

Another way to state the same observation is that one can view both Dirichlet and Neumann boundary conditions as limits of Robin in mechanics and field theory, but in general relativity only the Neumann boundary condition can be viewed as a limit of Robin. At the technical level, the problem is that for γ, the key relation (25) holds, but for π there is no such relation.

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