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Nonclassical face of quantum decay processes

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Abstract. A detailed analysis of the quantum decay processes shows the survival probability $\mathcal{P}(t)$ can not take that exponential form at any time interval including times smaller than the lifetime τ. We show that for times $t \sim \tau$ and for the times later than τ the form of $\mathcal{P}(t)$ looks as a composition of an oscillating and exponential functions. The amplitude of these oscillations is very small for $t \ll \tau$ and grows with increasing time and depends on the model considered. We also study the survival probability of moving relativistic unstable particles with definite momentum $\vec{p} \neq 0$: It turns out that late time deviations of the survival probability of these particles from the exponential-like form of the decay law should occur much earlier than it follows from the classical standard approach resolving itself into replacing time t by t/γ (where γ is the relativistic Lorentz factor) in the formula for the survival probability $\mathcal{P}(t)$.

1. Introduction

Since the discovery of radioactivity and the formulation of the radioactive decay law by Rutheford and Sody [1] there is a conviction that the decay law has the exponential form, $N(t) = N_0 \exp[-\lambda t]$, where $\lambda > 0$ is a constant, $N(t)$ is the number of atoms of the radioactive element at the instant $t \geq 0$ and $N_0 = N(0)$. Wesisskopf–Wigner theory of spontaneous emission [2] has extended this belief on the quantum decay processes: They found that to a good approximation the quantum mechanical non–decay probability of the exited atomic levels is a decreasing function of time having exponential form. Further theoretical studies of the quantum decay process [3, 4] showed that such processes seem to have three stages: the early time (initial), exponential (or "canonical"), and late time having inverse–power law form [5]. Results of these theoretical studies were the reason that there is rather widespread belief that a universal feature of the quantum decay process is the presence of these three time regimes. In this situation, each experimental evidence of oscillating decay curve at times of the order of life times is considered as an anomaly: The so–called GSI–anomaly [6, 7] is an example. The question arises, if indeed in the case of one component quantum unstable systems such oscillations of the decay process at the "exponential" regime are an anomaly. The another question is: Whether and how such a possible oscillations depend on the motion of the unstable quantum system. So we need also to know how to describe the decay process of unstable quantum systems in motion.

Studying text books one finds that if the decay law of the unstable particle in rest has the following form $\mathcal{P}(t) = \exp[-\frac{\varGamma_0 t}{\hbar}] \equiv \mathcal{P}_c(t)$, then in the case of the moving particle with momentum $p \neq 0$ its decay law looks as follows $\mathcal{P}(t) \equiv \mathcal{P}(t;p) = \exp[-\frac{\varGamma_0 t}{\hbar \gamma}] = \mathcal{P}_c(t/\gamma)$, where $\varGamma_0$ is the decay rate (time t and $\varGamma_0$ are measured in the rest reference frame of the particle) and γ is the relativistic Lorentz factor, $\gamma \equiv 1/\sqrt{1-\beta^2}$, $\beta = v/c$, $v = |\vec{v}|$ is the velocity of the

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particle, $\vec{v} = c\vec{p}/\sqrt{\vec{p}^2 + m_0^2\,c^2}$ and m_0 – is the rest mass. People are convinced that this equality being classical physics relation is valid also for any t in the case of quantum decay processes. The problem seems to be extremely important because from some theoretical studies it follows that in the case of quantum decay processes this relation is valid to a sufficient accuracy only for not more than a few lifetimes $\tau_0 = \hbar/\varGamma_0$ [8, 9, 10]. All these problems require a deeper analysis, elements of which will be presented below.

2. Properties of quantum unstable systems

Quantum unstable systems are characterized by their survival probability. We will assume that the reference frame $\mathcal{O}_0$ is the common inertial rest frame for the observer and for the unstable system. Now if the system in the rest frame is in the initial unstable state $|\phi\rangle \in \mathcal{H}$, ($\mathcal{H}$ is the Hilbert space of states of the considered system) prepared at the initial instant $t_0 = 0$, then the survival probability (the decay law), $\mathcal{P}_0(t)$, of the state $|\phi\rangle$ decaying in vacuum is given by the following formula $\mathcal{P}_0(t) = |a_0(t)|^2$, where $a_0(t)$ is the probability amplitude of finding the system at the time t in the rest frame $\mathcal{O}_0$ in the initial unstable state $|\phi\rangle$, $a_0(t) = \langle\phi|\phi(t)\rangle$. and $|\phi(t)\rangle$ is the solution of the Schrödinger equation for the initial condition $|\phi(0)\rangle = |\phi\rangle$,

$$i\frac{\partial}{\partial t}|\phi(t)\rangle = H|\phi(t)\rangle, \tag{1}$$

(We use units $\hbar = c = 1$). Here $|\phi\rangle, |\phi(t)\rangle \in \mathcal{H}$, and H is the total self–adjoint Hamiltonian of the system considered. We have $|\phi(t)\rangle = U(t)|\phi\rangle$, where $U(t) = \exp{[-itH]}$ is unitary evolution operator and $U(0) = \mathbb{I}$ is the unit operator.

The rest reference frame $\mathcal{O}_0$ is defined using common eigenvectors of H and of the momentum operator $\mathbf{P}$:

$$\mathbf{P}|\mu;p\rangle = \vec{p}|\mu;p\rangle, \quad \text{and,} \quad H|\mu;p\rangle = E'(\mu,p)\,|\mu;p\rangle, \tag{2}$$

where $\mu \equiv E'(\mu,0) \in \sigma_c(H)$ and $\sigma_c(H)$ is the continuous part of the spectrum of H. Operators H and $\mathbf{P}$ act in the state space $\mathcal{H}$. We have (see [8, 9, 11, 12]),

$$E'(\mu,p) \equiv \sqrt{\mu^2 + (\vec{p})^2}. \tag{3}$$

One obtains the rest reference frame $\mathcal{O}_0$ of the quantum unstable system assuming that $\vec{p} = 0$. Then $|\mu;0\rangle = |\mu;p=0\rangle$ and

$$\mathbf{P}|\mu;0\rangle = 0, \qquad H|\mu;0\rangle = \mu\,|\mu;0\rangle, \tag{4}$$

where $\mu \in \sigma_c(H)$. Eigenvectors $|\mu;0\rangle$ are normalized as usual: $\langle 0;\mu|\mu';0\rangle = \delta(\mu - \mu')$.

The unstable system in the rest frame $\mathcal{O}_0$ is modeled as the following wave–packet $|\phi_0\rangle \equiv |\phi_{\vec{p}=0}\rangle \stackrel{\text{def}}{=} |\phi\rangle$,

$$|\phi_0\rangle \equiv |\phi\rangle = \int_{\mu_0}^{\infty} c(\mu)\,|\mu;0\rangle\,d\mu, \tag{5}$$

where μ_0 is the lower bound of the spectrum $\sigma_c(H)$. The expansion coefficients $c(\mu)$ are functions of the mass parameter μ, (i.e. of the rest mass μ). The state $|\phi_0\rangle$ is normalized as follows:

$$\int_{\mu_0}^{\infty} |c(\mu)|^2\,d\mu = 1. \tag{6}$$

The expansion (5) and relation (4) allow one to find the amplitude $a_0(t)$ [4, 13],

$$a_0(t) \equiv \int_{\mu_0}^{\infty} \omega(\mu)\,e^{-\,i\,\mu\,t}\,d\mu, \tag{7}$$

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where $\omega(\mu) \equiv |c(\mu)|^2 > 0$.

Note that the use of the Schrödinger equation (1) allows one to find that within the problem considered.

$$i\frac{\partial}{\partial t}\langle\phi|\phi(t)\rangle = \langle\phi|H|\phi(t)\rangle. \tag{8}$$

This relation leads to the conclusion that the amplitude $a_0(t)$ satisfies the following equation

$$i\frac{\partial a_0(t)}{\partial t} = h(t)\, a_0(t), \tag{9}$$

where

$$h(t) = \frac{\langle\phi|H|\phi(t)\rangle}{a_0(t)}, \tag{10}$$

and $h(t)$ is the effective Hamiltonian governing the time evolution in the subspace of unstable states $\mathcal{H}_{||} = P\mathcal{H}$, where $P = |\phi\rangle\langle\phi|$ (see [14] and also [15, 16] and references therein). The subspace $\mathcal{H} \ominus \mathcal{H}_{||} = \mathcal{H}_{\perp} \equiv Q\mathcal{H}$ is the subspace of decay products. Here $Q = \mathbb{I} - P$. If $\langle\phi|H|\phi\rangle$ exists then using unitary evolution operator $U(t)$ and projection operators P and Q the relation (10) can be rewritten as follows

$$h(t) = \langle\phi|H|\phi\rangle \; + \; \frac{\langle\phi|HQ\,U(t)|\phi\rangle}{a_0(t)}. \tag{11}$$

This effective Hamiltonian $h(t)$ can be also derived starting from the Schrödinger equation for the total state space $\mathcal{H}$ and and looking for the rigorous evolution equation for a distinguished subspace of states $\mathcal{H}_{||} \subset \mathcal{H}$ [14].

In general in the case of one–dimensional $\mathcal{H}_{||}$ there is

$$h(t) = \mu_{\phi}(t) - \frac{i}{2}\gamma_{\phi}(t), \tag{12}$$

and $\mu_{\phi}(t) = \Re\,[h(t)]$, $\gamma_{\phi}(t) = -\,2\,\Im\,[h(t)]$, are the instantaneous mass (energy) $\mu_{\phi}(t)$ and the instantaneous decay rate, $\gamma_{\phi}(t)$, [14, 15, 16]. The relations (9) and (10) are convenient when the density $\omega(\mu)$ is given and one wants to find the instantaneous mass $\mu_{\phi}(t)$ and decay rate $\gamma_{\phi}(t)$: Inserting $\omega(\mu)$ into (7) one obtains the amplitude $a_0(t)$ and then one can find $h(t)$ and thus $\mu_{\phi}(t)$ and $\gamma_{\phi}(t)$.

Note that the state vector $|\phi\rangle$ of the form (5) describing a quantum unstable system can not be an eigenvector of the Hamiltonian H, otherwise it would be that $\mathcal{P}_0(t) = |\langle\phi|\phi(t)\rangle|^2 = |\langle\phi|\exp[-itH]\phi\rangle|^2 \equiv 1$ for all times t. So the mass (energy) of such a system is not defined. Simply the mass can not take the exact constant value in this state $|\phi\rangle$. In such a case quantum systems are characterized by the mass (energy) distribution density $\omega(\mu)$ and the average mass $< m >= \int_{\mu_0}^{\infty} \mu\,\omega(\mu)\,d\mu$ or by the instantaneous mass (energy) $\mu_{\phi}(t)$ but not by the exact value of the mass.

As it was mentioned in Sec. 1 one of the aims of these considerations is to check whether the decay law $\mathcal{P}_0(t)$ may be of the pure exponential form at some time intervals. The simplest way to compare the decay law $\mathcal{P}_0(t)$ with the exponential (canonical) decay law $\mathcal{P}_c(t)$, where $\mathcal{P}_c(t) = |a_c(t)|^2$ and

$$a_c(t) = \exp\,[-i\frac{t}{\hbar}(m_{\phi} - \frac{i}{2}\mathit{\Gamma}_{\phi}], \tag{13}$$

is to analyze properties of the following function [17]:

$$\zeta(t) \stackrel{\text{def}}{=} \frac{a_0(t)}{a_c(t)}. \tag{14}$$

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It is because $|\zeta(t)|^2 = \frac{\mathcal{P}_0(t)}{\mathcal{P}_c(t)}$. One can easy find that

$$\frac{\partial \zeta(t)}{\partial t} = \frac{i}{\hbar}\,(m_\phi - \frac{i}{2}\,\mathit{\Gamma}_\phi)\,\zeta(t)\; - \frac{i}{\hbar}\,h(t)\,\zeta(t), \tag{15}$$

where $h(t)$ is the effective Hamiltonian defined by relations (9), (10). Let us assume now that $\langle\phi|H|\phi\rangle$ exists and such instants $0 < t_1 < t_2 < \infty$ of time t exist that for any $t \in (t_1, t_2)$ there is

$$\zeta(t) = \zeta(t_1) = \zeta(t_2) = const \stackrel{\text{def}}{=} c_\phi \neq 0. \tag{16}$$

It can occur only if $\frac{\partial \zeta(t)}{\partial t} = 0$ for all $t \in (t_1, t_2)$. By definition $\zeta(t) \neq 0$ and therefore from (15) we conclude that it is possible only and only if

$$h(t)\; - \;(m_\phi - \frac{i}{2}\,\mathit{\Gamma}_\phi) = 0, \quad \text{for } \; t_1 \leq t \leq t_2, \tag{17}$$

that is, if and only if

$$h(t_1) = h(t) = h(t_2) = const \stackrel{\text{def}}{=} c_h \neq 0 \quad \text{for } \; t_1 \leq t \leq t_2. \tag{18}$$

From (11) the conclusion follows that the equality $h(t_1) = h(t) = c_h$ can take place if

$$a_0(t)\,\langle\phi|HQ\,U(t_1)|\phi\rangle = a_0(t_1)\,\langle\phi|HQ\,U(t)|\phi\rangle. \tag{19}$$

We have that $a_0(t) \neq 0,\; a_0(t_1) \neq 0$ and $|a(t)| < \infty$. Hence $\lambda(t, t_1) \stackrel{\text{def}}{=} a_0(t)/a_0(t_1)$ is a complex function such that $0 < |\lambda(t, t_1)| < \infty$. Using this $\lambda(t, t_1)$ one can replace the relation (19) by

$$\langle\phi|HQ\,U(t_1)\,\Big[\,\lambda(t, t_1)|\phi\rangle\; - \;W(t, t_1)|\phi\rangle\Big] = 0, \tag{20}$$

where $W(t, t_1) = U^{+}(t_1)\,U(t)$ is the unitary operator.

As it is seen the condition (20) can be satisfied in two cases: The first one is

$$W(t, t_1)|\phi\rangle\; - \;\lambda(t, t_1)|\phi\rangle \;=\; 0, \tag{21}$$

and the second one takes place when $[\,\lambda(t, t_1)|\phi\rangle\; - \;W(t, t_1)|\phi\rangle] \neq 0$, and

$$(\langle\phi|H)^{+} = H|\phi\rangle \quad \perp \quad Q\,U(t_1)\,[\lambda(t, t_1)|\phi\rangle\; - \;W(t, t_1)|\phi\rangle]. \tag{22}$$

The first case implies that $h(t_1) = h(t) = c_h = const$ (which by (18) means that $\frac{\partial \zeta(t)}{\partial t} = 0$) if and only if the vector $|\phi\rangle$ representing an unstable state of the system is an eigenvector for the unitary operator $W(t, t_1)$. On the other hand if the condition (16) is satisfied then,

$$\lambda(t, t_1) = \frac{a_0(t)}{a_0(t_1)} \equiv \frac{a_0(t)}{a_c(t)}\,\frac{a_c(t)}{a_0(t_1)} \equiv \exp\,[\frac{i}{\hbar}(m_\phi - \frac{i}{2}\mathit{\Gamma}_\phi)(t - t_1)], \tag{23}$$

and thus for $t > t_1$ one finds that in such a case there must be $|\lambda(t, t_1)| < 1$. This conclusion means that the equation (21) has no solution when the condition (16) holds: Eigenvalues $\lambda(t, t_1)$ of any unitary operator must satisfy the condition $|\lambda(t, t_1)| = 1$.

Let us consider now the second case: The definition of the projectors P and Q suggests that this case can be realized only if the vector $H|\phi\rangle$ is proportional to the vector $|\phi\rangle$: $H|\phi\rangle = \alpha_\phi|\phi\rangle$. So $\frac{\partial \zeta(t)}{\partial t} = 0$ if and only if the vector $|\phi\rangle$ representing the unstable state of the system considered is an eigenvector for the total Hamiltonian H, which is in clear contradiction with the condition that the vector $|\phi\rangle$ representing the unstable state cannot be the eigenvector for H.

Implications of the above conclusions for possible realizations of the relation (20) allow one to deduce that the supposition that such time interval $[t_1, t_2]$ can exist that $h(t_1) = h(t) = c_h = const$ for $t \in (t_1, t_2)$ and thus $\zeta(t) = const = \zeta(t_1) = \zeta(t_2)$ for $t \in (t_1, t_2)$ is false. Hence from the definition of $\zeta(t)$ the conclusion follows: Within the approach considered in this paper for any time interval $[t_1, t_2]$ the decay law can not be described by the exponential function of time. This conclusion is the general one. It does not depend on models of quantum unstable states.

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3. The Breit–Wigner (BW) model

In this part of the paper analytical results presented in the previous section will be illustrated graphically. The typical form of the survival probability $\mathcal{P}_0(t)$ is shown in Fig (1). The calculations were made for the distribution of the mass (energy) density $\omega(\mu)$ having the Breit–Wigner (BW) form $\omega(\mu) \equiv \omega_{BW}(\mu)$,

$$\omega_{BW}(\mu) = \frac{N}{2\pi}\,\Theta(\mu - \mu_0)\,\frac{\Gamma_0}{(\mu - m_0)^2 + (\frac{\Gamma_0}{2})^2}, \tag{24}$$

where N is a normalization constant and $\Theta(\mu)$ is a step function: $\Theta(\mu) = 0$ for $\mu \leq 0$ and $\Theta(\mu) = 1$ for $\mu > 0$. The form of the decay curves depend on the ratio $s_R = \frac{m_R}{\Gamma_0}$, where $m_R = m_0 - \mu_0$: The smaller s_R, the shorter the time when the late time deviations from the exponential form of $\mathcal{P}_0(t)$ begin to dominate.

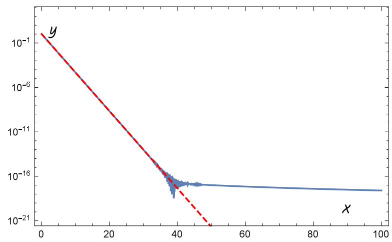

Figure 1. Decay curves obtained for $\omega_{BW}(E)$ given by Eq. (24). Axes: $x = t/\tau_0$ — time t is measured in lifetimes τ_0, y — survival probabilities on a logarithmic scale (The solid line: the decay curve $\mathcal{P}_0(t) = |a_0(t)|^2$; The dotted line: the canonical decay curve $\mathcal{P}_c(t) = |a_c(t)|^2$). The case $s_R = \frac{m_R}{\Gamma_0} = 1000$.

Within the considered model the standard canonical form of the survival amplitude $a_c(t)$ is given by formula (13) with m_0 and Γ_0 replacing m_ϕ and Γ_ϕ, (where Γ_0 is the decay rate and $\frac{\hbar}{\Gamma_0} \equiv \frac{1}{\Gamma_0} = \tau_0$ is the lifetime within the assumed system of units $\hbar = c = 1$: Here time t and Γ_0 are measured in the rest reference frame of the particle), and thus $\mathcal{P}_c(t) = |a_c(t)|^2 \equiv \exp\,[-\frac{\Gamma_0}{\hbar}\,t]$, is the canonical form of the survival amplitude.

The situation when $\omega(\mu) = \omega_{BW}(\mu)$ is the typical case discussed in numerous papers and used therein to model decay processes. Hence the need for analysis of the real form of the decay curves obtained using $\omega(\mu) = \omega_{BW}(\mu)$ and this is why we consider this case here.

As already noted the function $\zeta(t)$ (see (14)) is very helpful in achieving this goal. Analysis of properties of $|\zeta(t)|^2$ allows one to visualize all the more subtle differences between $\mathcal{P}_0(t)$ and $\mathcal{P}_c(t)$. The function $\zeta(t)$ was used to find numerically $|\zeta(t)|^2$ for $\omega(m) = \omega_{BW}(m)$. Results of numerical calculations are presented in Figs (2) and (3): One can see that in the case considered the form of $|\zeta(t)|^2$ also depend on the ratio $s_R \stackrel{\text{def}}{=} \frac{m_R}{\Gamma_0} \equiv \frac{m_0 - \mu_0}{\Gamma_0}$ [17].

From results presented in Figs (2), (3) one can see that the oscillating decay curves of one component unstable system can not be considered as something extraordinary or as anomaly: It seems to be a universal feature of the decay process [17]. Results of the model calculations presented in Figs (2) and (3) shows that at the initial stage of the "exponential" (or "canonical") decay regime the amplitude of these oscillations may be much less than the accuracy of detectors. Then with increasing time the amplitude of oscillations grows (see Fig. (3)), which increases the chances of observing them. So the conclusion that this is a true quantum picture of the decay process at the so–called "exponential" regime of times seems to be justified.

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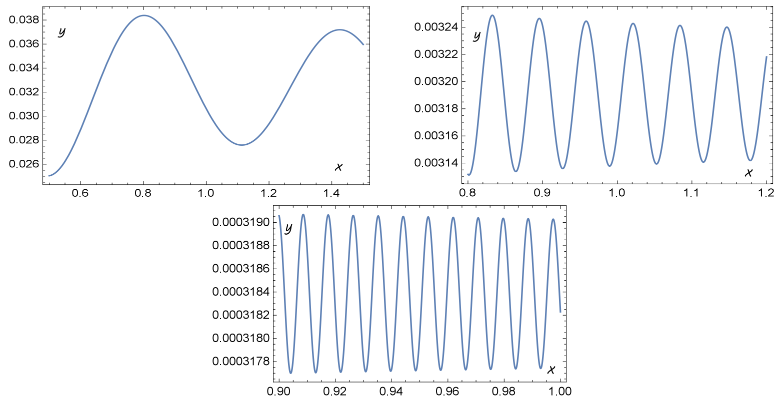

Figure 2. A comparison of decay curves obtained for $\omega_{BW}(\mu)$ given by Eq. (24) with canonical decay curves. Axes: $x = t/\tau_0$ — time t is measured in lifetimes τ_0, y — The function $f(t) = (|\zeta(t)|^2 - 1) = \frac{\mathcal{P}_0(t)}{\mathcal{P}_c(t)} - 1$, where $\zeta(t)$ is defined by the formula (14). **Left panel**: $s_R = 10$. **Right panel**: $s_R = 100$. **Lower panel**: $s_R = 1000$.

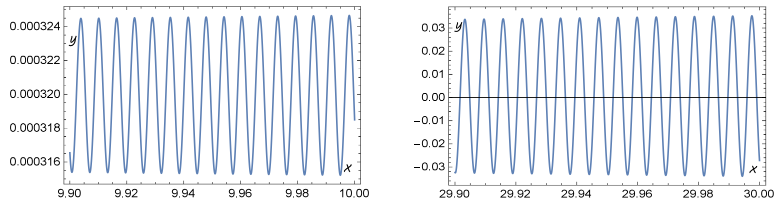

Figure 3. A comparison of decay curves obtained for $\omega_{BW}(\mu)$ given by Eq. (24) with canonical decay curves. **Both panels** – Axes: $x = t/\tau_0$ — time t is measured in lifetimes τ_0, y — The function $f(t) = (|\zeta(t)|^2 - 1) = \frac{\mathcal{P}_0(t)}{\mathcal{P}_c(t)} - 1$, where $\zeta(t)$ is defined by the formula (14), $\mathcal{P}_0(t) = |a_0(t)|^2$, $\mathcal{P}_c(t) = |a_c(t)|^2$. The case $s_R = 1000$.

4. Moving quantum unstable systems

Let us analyze now some properties of the moving quantum unstable systems. In this case we need the probability amplitude $a_p(t) = \langle\phi_p|\phi_p(t)\rangle$, which defines the survival probability $\mathcal{P}_p(t) = |a_p(t)|^2$. We have $|\phi_p(t)\rangle \stackrel{\text{def}}{=} \exp\left[-itH\right]|\phi_p\rangle$ in $\hbar = c = 1$ units. As it is seen we need the vector $|\phi_p\rangle$ and eigenvalues $E'(\mu, p)$ solving Eq. (2). Vectors $|\phi\rangle$ which is defined by the relation (5) and $|\phi_p\rangle$ are elements of the same state space $\mathcal{H}$ connected with the coordinate rest system of the observer $\mathcal{O}$: We are looking for the decay law of the moving particle measured by the observer $\mathcal{O}$. If to assume for simplicity that $\mathbf{P} = (P_1, 0, 0)$ and that $\vec{v} = (v_1, 0, 0) \equiv (v, 0, 0)$ then there is $\vec{p} = (p, 0, 0)$ for the eigenvalues $\vec{p}$ of the momentum operator $\mathbf{P}$ and $|\vec{p}| = p$. Hence (see [8, 9, 11, 12]),

$$H|\mu; p\rangle = \sqrt{\mu^2 + p^2}\,|\mu; p\rangle \tag{25}$$

which replaces Eq. (2).

We can model the moving quantum unstable particle ϕ with constant momentum, $\vec{p}$, analogously as the quantum unstable system in the rest frame (when $\vec{p} = 0$), that is as the following wave–packet $|\phi_p\rangle$,

$$|\phi_p\rangle = \int_{\mu_0}^{\infty} c(\mu)\,|\mu; p\rangle\, dm. \tag{26}$$

Here coefficients $c(\mu)$ are functions of the mass parameter μ, (in other words of the rest mass μ), which is Lorentz invariant and therefore the scalar functions $c(\mu)$ of μ are also Lorentz invariant and are the same as in the rest reference frame $\mathcal{O}_0$.

The final relation for for the amplitude $a_p(t)$ results from (25) and form the equation (26), (see [8, 9, 12]),

$$a_p(t) \;\; \equiv \;\; \int_{\mu_0}^{\infty} \omega(\mu) \, e^{-\,i\sqrt{\mu^2 + p^2}\;t} \, d\mu. \tag{27}$$

Results of numerical calculations are presented in Figs (4), (5), where calculations were performed for $\omega(\mu) = \omega_{BW}(\mu)$ and $\mu_0 = 0$, $E_0/\varGamma_0 \equiv m_0/\varGamma_0 = 1000$ and $cp/\varGamma_0 \equiv p/\varGamma_0 = 1000$. Values of these parameters correspond to $\gamma = \sqrt{2}$, which is very close to γ from the experiment performed by the GSI team [6, 7] and this is why such values of them were chosen in our considerations. According to the literature for laboratory systems a typical value of the ratio $m_0/\varGamma_0$ is $m_0/\varGamma_0 \geq O(10^3 - 10^6)$ (see eg. [18]) therefore the choice $m_0/\varGamma_0 = 1000$ seems to be reasonable minimum. Decay curves obtained numerically are presented in Fig (4) (see also [10]).

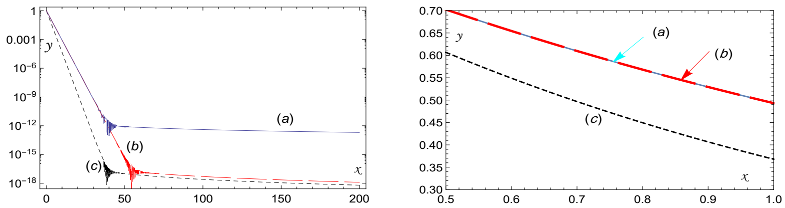

Figure 4. Decay curves obtained for $\omega_{BW}(m)$ given by Eq. (24). Axes: $x = t/\tau_0$ — time t is measured in lifetimes τ_0, y — survival probabilities (**Left panel**: the logarithmic scales, (a) the decay curve $\mathcal{P}_p(t)$, (b) the decay curve $\mathcal{P}_0(t/\gamma)$, (c) the decay curve $\mathcal{P}_0(t)$; **Right panel**: (a) – $\mathcal{P}_p(t)$, (b) – $\mathcal{P}_0(t/\gamma)$, (c) – $\mathcal{P}_0(t)$).

The ratio $\mathcal{P}_p(t)/\mathcal{P}_c(t/\gamma)$ in the case of moving particles can be also calculated similarly as it was in the case of quantum unstable systems in rest. Results of numerical calculations of this ratio are presented in Fig. (5), and calculations were performed for $\omega(\mu) = \omega_{BW}(\mu)$ and for $\mu_0 = 0$, $m_0/\varGamma_0 = 1000$, $cp/\varGamma_0 \equiv p/\varGamma_0 = 1000$ and $\gamma = \sqrt{2}$.

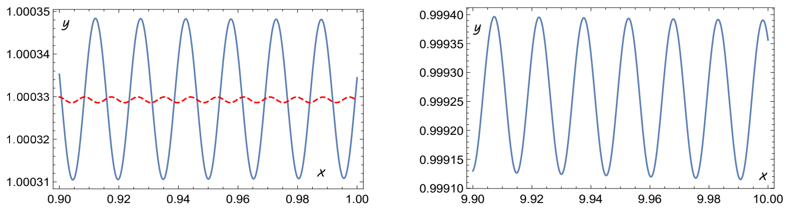

Figure 5. Left panel. Axes: $x = t/\tau_0$ — time t is measured in lifetimes τ_0, y — Ratio of probabilities — Solid line: $\mathcal{P}_p(t)/\mathcal{P}_c(t/\gamma)$; Dashed line $\mathcal{P}_0(t/\gamma)/\mathcal{P}_c(t/\gamma)$. **Right panel**. Axes: $x = t/\tau_0$ — time t is measured in lifetimes τ_0, y — Ratio of probabilities: $\mathcal{P}_p(t)/\mathcal{P}_c(t/\gamma)$.

5. Summary

As it was pointed out earlier the mass (energy) of the system in the unstable state $|\phi\rangle$ is not defined: It can not take the exact value. Unstable system can be characterized by the mass distribution $\omega(\mu)$, the average mass $< m >= \int_{\mu_0}^{\infty} \mu\, \omega(\mu) d\mu$ and by instantaneous mass (energy) $\mu_\phi(t)$ but not by the mass. Also it was shown that there is no any time interval in which the survival probability (decay) law could be a decreasing function of time of the purely exponential form: Even in the case of the BW model in so–called "exponential regime" the decay curves are oscillatory modulated with smaller or large amplitude of oscillations depending on the parameters of the model. The another observation was that in the case of moving relativistic quantum unstable system moving with constant momentum $\vec{p}$, when unstable systems are modeled by the BW mass distribution $\omega_{BW}(\mu)$, only at times of the order of lifetime τ_0 it can be $\mathcal{P}_p(t) \simeq \mathcal{P}_0(t/\gamma)$ to a better or worse approximation. At times longer than a few lifetimes the decay process of moving particles observed by an observer in his rest system is much slower that it follows from the classical physics relation $\mathcal{P}_p(t) \stackrel{?}{=} \exp[-\frac{t}{\gamma}\, \varGamma_0]$: There is $\mathcal{P}_p(t) > \mathcal{P}_0(t/\gamma), \quad \text{for} \quad t \gg \tau_0$. Therefore there is a need to test the decay law of moving relativistic unstable system for times much longer than the lifetime. It was shown also that in the case of moving relativistic quantum unstable system moving with constant momentum $\vec{p}$ decay curves are also oscillatory modulated but the amplitude of these oscillations is higher than in the case of unstable systems in rest.

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