

# Probability of boundary conditions in quantum cosmology

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**Abstract.** One of the main interest in quantum cosmology is to determine boundary conditions for the wave function of the universe which can predict observational data of our universe. For this purpose, we solve the Wheeler-DeWitt equation for a closed universe with a scalar field numerically and evaluate probabilities for boundary conditions of the wave function of the universe. To impose boundary conditions of the wave function, we use exact solutions of the Wheeler-DeWitt equation with a constant scalar field potential. We specify the exact solutions by introducing two real parameters to discriminate boundary conditions, and obtain the probability for these parameters under the requirement of sufficient e-foldings of the inflation. The probability distribution of boundary conditions prefers the tunneling boundary condition to the no-boundary boundary condition. Furthermore, for large values of a model parameter related to the inflaton mass and the cosmological constant, the probability of boundary conditions selects an unique boundary condition different from the tunneling type.

## 1. Introduction

Investigation of the very early period of the universe requires quantum treatment of gravity [1]. However, because we do not have the complete theory of quantum gravity yet in hand, simplified models with reduced dynamical degrees of freedom have been investigated to understand nature of canonical quantum gravity. This approach is the mini-superspace quantum cosmology (a general review of quantum cosmology is given by [2]). A quantum state of the model is represented by the wave function of the universe, which satisfies the Wheeler-DeWitt (WD) equation derived from the procedure of canonical quantization [3]. The wave function of the universe is represented as the path integral by summing over histories of the universe [4].

To obtain the wave function of the universe, we must impose boundary conditions of the WD equation. In the context of the quantum cosmology, there are two major candidates for the boundary condition, the tunneling proposal by Vilenkin [5, 6] and the no-boundary boundary condition proposal by Hartle and Hawking [7]. The former is given by the wave function only consisting of the outgoing mode at the asymptotic future of mini-superspace, and is analogous to the tunneling wave function in quantum mechanics. The latter is given by the path integral over Euclidean non-singular compact geometries with no-boundary. In the path integral representation of the wave function, a choice of the path integral contour corresponds to specifying a boundary condition of the WD equation [8, 9]. To predict the classical universe using quantum cosmology, we must derive a probability for classical observables from the wave function of the universe. The number of e-foldings of inflation is often used as a predictable observable and recent observational restriction requires this number must be greater than about 60. The amount of e-foldings predicted by quantum cosmology depends on models and boundary conditions. Thus, the main issue in quantum cosmology is to determine which type of boundary conditions are preferable to explain observational results.

In this research, we apply a numerical method to obtain predictions from wave functions of the universe. The main idea of our research is to represent boundary conditions of the wave

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function using exact solutions of the WD equation with a constant scalar field potential. This makes our problem of determining boundary conditions as the parameter estimation in space of boundary conditions. We aim to obtain a probability distribution of boundary conditions under the constraint of sufficient e-foldings of the inflation.

## 2. Mini-superspace model

### 2.1. Classical model and quantization

We consider the Einstein gravity with a cosmological constant and a minimally coupled massive scalar field as the inflaton. The action of the total system is given by

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} (R - 2\Lambda) - \frac{1}{2} \int d^4x \sqrt{-g} [(\partial_\mu \Phi)^2 + m^2 \Phi^2]. \quad (1)$$

We assume a homogeneous and isotropic closed universe. Then the geometry of the universe is given by the Friedmann-Robertson-Walker (FRW) metric with a scale factor. The metric is given by

$$ds^2 = \frac{3}{\Lambda} \left( -\frac{N^2}{q} d\lambda^2 + q d\Omega_3^2 \right), \quad (2)$$

where  $\lambda$  is a dimensionless time parameter and  $N$  is a lapse function. We introduce a dimensionless field variable  $\phi = (4\pi G/3)^{1/2} \Phi$  and its mass  $\mu = (3/\Lambda)^{1/2} m$ . In this model, dynamical variables are the scale factor  $q(\lambda)$  and the inflaton field  $\phi(\lambda)$ . We represent them as coordinates of configuration space  $q^A = (q^0, q^1) = (q, \phi)$ . This configuration space is called mini-superspace. The total Hamiltonian of our mini-superspace model is given by

$$H_T = \frac{KN}{2} \left[ \frac{1}{K^2} \left( -4p_q^2 + \frac{1}{q^2} p_\phi^2 \right) - 1 + q(1 + \mu^2 \phi^2) \right] = NH. \quad (3)$$

where we introduced a constant  $K \equiv 9\pi/(2G\Lambda)$ . We obtain the Hamiltonian constraint  $H = 0$  by taking variation of the lapse function  $N$ . Canonical quantization of the model is performed by replacing  $p_A$  in the Hamiltonian constraint by differential operator  $\hat{p}_A = -i\partial/\partial q^A$ , and imposing operator version of the Hamiltonian constraint  $\hat{H}$  on a physical state  $\Psi(q^A)$ . It yields the Wheeler-DeWitt equation

$$\left[ -\frac{1}{2K^2} G^{AB} \partial_A \partial_B + U(q) \right] \Psi(q) = 0, \quad U = -\frac{1}{2} + qV, \quad G^{AB} = \text{diag}(-4, 1/q^2). \quad (4)$$

The wave function  $\Psi(q^A)$  on mini-superspace is called the wave function of the universe. Although there is an operator ordering ambiguity in the quantization procedure, we choose the ordering which yields the equation (4) in our analysis.

### 2.2. Semi-classical approximation and probability

To extract predictions for the classical universe from the wave function, it must be expressed as the semi-classical form, which means the wave function behaves as the WKB solution of Eq. (4). We perform the WKB expansion of the wave function as  $\Psi(q^A) = \sum_i C^{(i)}(q^A) e^{-\frac{1}{\hbar} I^{(i)}(q^A)}$  where sum is taken over the independent WKB components. For convergence of the path integral representation of the wave function, the contour of the path integral must be analytically continued in the complex plane. Thus,  $I$  and  $C$  would generally become complex functions. We write  $I$  as  $I = I_R - iS$  where  $I_R$  and  $S$  are real functions. From the WD equation (4), we obtain a set of semi-classical equations for  $I, C$ :

$$O(\hbar^0): -\frac{1}{2K^2} (\nabla I)^2 + U(q^A) = 0, \quad O(\hbar^1): 2\nabla I \cdot \nabla C + C \nabla^2 I = 0, \quad (5)$$

where  $(\nabla I)^2 = G^{AB} \partial_A I \partial_B I$ ,  $(\nabla I) \cdot (\nabla C) = G^{AB} \partial_A I \partial_B C$ ,  $\nabla^2 I = G^{AB} \partial_A \partial_B I$ . Now, we consider a condition of the wave function to provide predictions for the classical universe. In the classical regime, the phase of the wave function must satisfy the Hamilton-Jacobi equation. This implies that the equation (5) corresponds to the Hamilton-Jacobi equation  $1/(2K^2)(\nabla S)^2 + U = 0$  in the classical regime. Thus  $I_R$  and  $S$  should satisfy the condition  $|\nabla I_R|^2/|\nabla S|^2 \ll 1$ . This inequality is called the ‘‘classicality’’ condition [10, 11]. To predict the evolution of classical universes from the wave function, we expect that the probability can be obtained in the region of mini-superspace where the classicality condition is satisfied.

For the wave function  $\Psi$  satisfying the WD equation, we have the conserved current in mini-superspace  $\mathcal{J}_A = (i/2)(\Psi^* \nabla_A \Psi - \Psi \nabla_A \Psi^*)$ ,  $\nabla \cdot \mathcal{J}_A = 0$ . In the classical region where the wave function has the WKB form, we can obtain the positive definite probability measure from  $\mathcal{J}_A$ . For each WKB components of the wave function, we define  $J_A^{(i)} \equiv -|C^{(i)}|^2 \exp(-2I_R^{(i)}) \nabla_A S^{(i)}$ . They are conserved independently in the classical region  $\nabla \cdot J^{(i)} = 0$ . From the Hamilton-Jacobi equation, we can assign the canonical momentum in the classical region  $p_A^{(i)} = \nabla_A S^{(i)} = \partial S^{(i)} / \partial q^A$ . Here, we focus only on the components with  $p_q^{(i)} = \partial_q S^{(i)} < 0$ . From  $dq^{(i)} / d\lambda \propto -p_q^{(i)} > 0$ , these components correspond to expanding universes. Thus, we can introduce a conserved current corresponding to expanding universes as  $J_A^+ \equiv -\sum_{p_q^{(i)} < 0} |C^{(i)}|^2 \exp(-2I_R^{(i)}) \nabla_A S^{(i)}$ . Let us consider a surface  $\Sigma_c$  in mini-superspace which is spacelike with respect to the metric  $G_{AB}$  and has a unit normal  $n_A$ . We require the classicality condition is satisfied on this surface. Then the relative probability  $\mathcal{P}(\Sigma_c)$  of classical histories passing through this surface is given by the component of the conserved current along the normal if it is positive. In the leading order in  $\hbar$ , this is  $\mathcal{P}(\Sigma_c) \equiv J^+ \cdot n = -\sum_{p_q^{(i)} < 0} |C^{(i)}|^2 \exp(-2I_R^{(i)}) \nabla_n S^{(i)}$  where  $\nabla_n$  means differentiation along the normal vector  $n_A$ . As a point on  $\Sigma_c$  is specified by the value of the scalar field,  $\mathcal{P}(\phi) \equiv \mathcal{P}(\Sigma_c(\phi))$  provides the probability for the inflaton field to realize a value  $\phi$  on  $\Sigma_c$ .

We can derive a probability for observables from the probability measure  $\mathcal{P}(\phi)$ . It can be given as the conditional probability [2]  $P(s_0|s_1) = \int_{s_0} J \cdot d\Sigma_c / \int_{s_1} J \cdot d\Sigma_c$ ,  $s_0 \subset s_1$  where  $s_1$  is a subset of the hypersurface  $\Sigma_c$  defined by some theoretical constraints and  $s_0$  is a subset of  $s_1$  defined by restricting  $s_1$  using observational constraints. By using the relation  $p_A = \partial S / \partial q^A$ , we can obtain classical trajectories starting from  $\Sigma_c$ . Namely, the probability measure on  $\Sigma_c$  with the classicality condition gives probability distribution of initial data  $(p_q, q, p_\phi, \phi)$  for the classical equation of motion. In our analysis, the number of e-foldings  $\mathcal{N} \equiv \log(a(t_f)/a(t_i))$  is adopted as an observable. We define  $t_f$  as the end time of inflation driven by the scalar field potential. The number of e-foldings  $\mathcal{N}$  is determined by the initial data and it is possible to translate the probability measure for  $\phi$  on  $\Sigma_c$  to the probability measure for  $\mathcal{N}(\phi)$ . To introduce the conditional probability, we define an interval  $s_1$  as  $s_1 = [\phi_{\min}, \phi_{\text{pl}}]$  where  $\phi_{\min}$  is the lower bound of the interval and  $\phi_{\text{pl}} = 4\sqrt{2K}/(3\mu)$  is the value of the inflaton field corresponding to the Planck energy density  $m_{\text{pl}}^4$ . Then an interval  $s_0 \subset s_1$  is defined as  $s_0 = [\phi_{\text{suf}}, \phi_{\text{pl}}]$  where  $\phi_{\text{suf}}$  corresponds to the number of e-foldings  $\mathcal{N}_{\text{suf}} \approx 60$  consistent with observations. Accordingly, the conditional probability to predict the universe with sufficient inflation becomes

$$P_{\text{suf}} \equiv P(s_0|s_1) = \frac{\int_{\phi_{\text{suf}}}^{\phi_{\text{pl}}} d\phi \mathcal{P}(\phi)}{\int_{\phi_{\min}}^{\phi_{\text{pl}}} d\phi \mathcal{P}(\phi)}. \quad (6)$$

These probability and expectation value depend not only on cosmological models but also on boundary conditions of the wave function. By calculating and comparing  $P_{\text{suf}}$  for given models and given boundary conditions, we can evaluate what type of models and boundary conditions are more suitable to explain observation of our universe.

wave function	parameter $(a, b)$	asymptotic form for $q \gg 1$
$\Psi_{\text{HH}}$	$(\pi/4, 0)$	$\sim \exp(+K/(6V)) \cos S_0$
$\Psi_V$	$(\pi/4, \pi/2)$	$\sim \exp(-K/(6V)) \exp(-iS_0)$
$\Psi_1$	$(\pi/2, \pi/2)$	$\sim \exp(-K/(6V)) \cos S_0$
$\Psi_2$	$(\pi/2, 0)$	$\sim \exp(+K/(6V)) \cos S_0$
$\Psi_3$	$(0, \text{any values})$	$\sim -\exp(-K/(6V)) \sin S_0$

**Table 1.** Typical wave functions and their parameters  $(a, b)$  and asymptotic behaviors. The phase function  $S_0$  is defined by  $S_0 = K/(6V)(2qV - 1)^{3/2} - \pi/4$ .

### 3. Probability for boundary conditions

When we have some restriction on our models of inflationary universe from observations, we can investigate a probability which states preferable type of boundary conditions. It is possible to express this probability using Bayes' theorem:

$$P(B_i|S) = \frac{P(B_i)P(S|B_i)}{\sum_k P(B_k)P(S|B_k)}, \quad (7)$$

where  $P(B_i|S)$  is a probability for  $B_i$  under  $S$  happened. Here,  $B_i$  is some candidate of a boundary condition of the wave function labeled by index  $i$ , and  $S$  means the universe with sufficient inflation, namely,  $\mathcal{N} \geq 60$ . Thus,  $P(B_i|S)$  denotes the probability for  $B_i$  under the sufficiently inflated universe. On the contrary,  $P(S|B_i)$  in the right hand side is the probability for sufficient inflation under the boundary condition  $B_i$  and is equivalent to  $P_{\text{suf}}$  defined in the previous section:  $P(S|B_i) = P_{\text{suf}}(B_i) = P(\mathcal{N} \geq 60|B_i)$ . As we do not have any information on the prior probability  $P(B_i)$ , we assume that it is uniformly distributed. To represent different boundary conditions, we will introduce two parameters  $a, b$  in (10). Then the probability for the parameters  $a, b$  is given by

$$P(a, b|S) = \frac{P(S|a, b)}{\int da' db' P(S|a', b')}. \quad (8)$$

When we solve the WD equation, we have to impose some boundary condition on the wave function. For this purpose, we use exact solutions of the WD equation which are obtained when the scalar field potential  $V(\phi)$  is constant. Based on the path integral representation of the wave function, for the constant scalar field potential case, the wave function corresponding to the no-boundary (Hartle-Hawking) and the tunneling (Vilenkin) type boundary conditions are expressed as [8]  $\Psi_{\text{HH}} = \Psi_2 + \Psi_3$ ,  $\Psi_V = \Psi_1 + i\Psi_3$ , where

$$\Psi_1 \equiv (2V)^{-1/3} \text{Ai}(z_0) \text{Ai}(z), \quad \Psi_2 \equiv (2V)^{-1/3} \text{Bi}(z_0) \text{Ai}(z), \quad \Psi_3 \equiv (2V)^{-1/3} \text{Ai}(z_0) \text{Bi}(z), \quad (9)$$

with  $z = z(q) = (4V/K)^{-2/3} (1 - 2qV)$ ,  $z_0 = z(0) = (4V/K)^{-2/3}$ . For large values of the scale factor (classical region),  $\Psi_{\text{HH}}$  is superposition of expanding and contracting universes with amplitude  $\exp(+K/(6V))$  which prefers small values of the potential. On the other hand,  $\Psi_V$  represents an expanding universe with amplitude  $\exp(-K/(6V))$  which prefers large values of the potential. We can express more general type of wave functions introducing two real parameters  $a, b$  which represent boundary conditions of the wave function

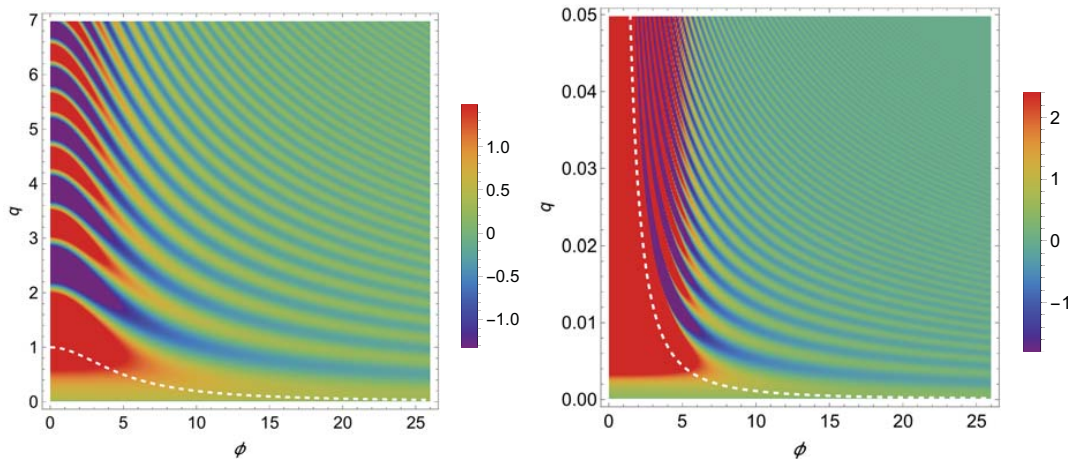
$$\Psi_C = \tan a (\cos b \Psi_2 - i \sin b \Psi_1) + \Psi_3, \quad 0 \leq a, b \leq \pi/2. \quad (10)$$

Introduced parameters  $a, b$  distinguish boundary conditions of the wave function (Table 1). Solving the wave function and calculating probability  $P_{\text{suf}}$  for different values of  $(a, b)$ , we can evaluate the probability for the parameters  $(a, b)$  using the relation (8).

#### 4. Numerical simulation of the wave function

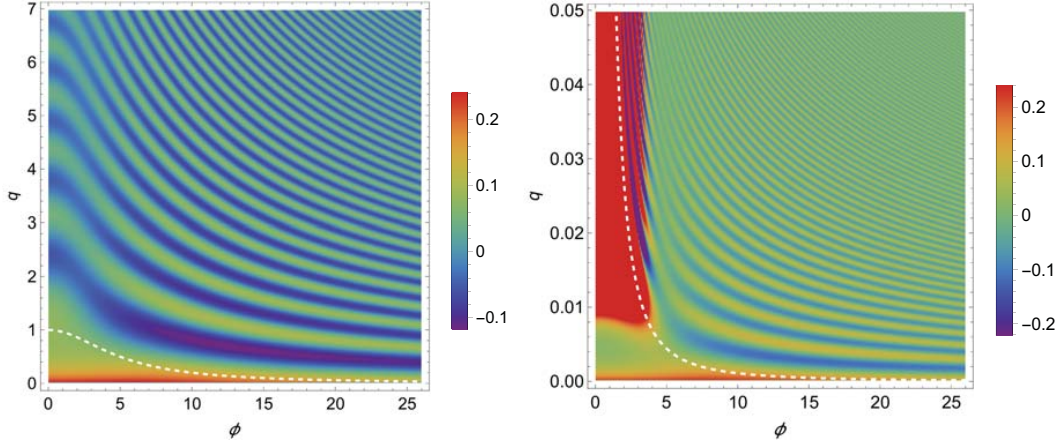
We solve the WD equation (4) numerically to obtain the probability of boundary conditions. The detail of our numerical method is explained in [12]. We prepare the initial surface  $q = q_{\text{ini}}$  in the Euclidean region of mini-superspace and impose the boundary condition  $\Psi(q_{\text{ini}}, \phi) = \Psi_C(q_{\text{ini}}, \phi)$ ,  $\partial_q \Psi(q_{\text{ini}}, \phi) = \partial_q \Psi_C(q_{\text{ini}}, \phi)$ . As  $\Psi_C$  introduced by (10) is specified by two parameters  $(a, b)$ , this boundary condition is also specified by these two parameters. We call  $\Psi_C$  the boundary wave function. For models with a constant scalar field potential, this boundary condition of course reproduces the exact solution  $\Psi_C$ . We fix the mass of the scalar field and consider two models with parameters  $\mu = 0.2$  ( $m^2 = 0.03$ ,  $\Lambda = 2.25$ ) and  $\mu = 3$  ( $m^2 = 0.03$ ,  $\Lambda = 0.01$ ). Different value of  $\mu$  corresponds to different value of the cosmological constant in our analysis. The former choice results in slow roll inflation followed by over damped rolling of the inflaton field and the later results in inflation with slow rolling followed by oscillation of the inflaton about  $\phi = 0$ . Our simulation algorithm is as follows:

- (i) Prepare an initial surface  $q = q_{\text{ini}}$  in the Euclidean region of mini-superspace close to  $q = 0$ .  $q_{\text{ini}}$  cannot be chosen too small because we must keep the Courant condition for stable numerical integration of the wave equation.
- (ii) Solve the WD equation numerically from  $q = q_{\text{ini}}$  to  $q_{\text{fin}}$  with a given boundary wave function  $\Psi_C$ . We adopt the 5-step Adams-Bashforth method for numerical integration which has the 5-th order accuracy. We used  $20000 \times 200$  grid size which covers  $q_{\text{ini}} \leq q \leq q_{\text{fin}}$ ,  $\phi_{\text{min}} \leq \phi \leq \phi_{\text{max}}$ .
- (iii) We specify a hypersurface  $\Sigma_c$  on which the classicality condition is satisfied. We choose  $\Sigma_c$  as a constant  $S_0$  surface. We numerically obtain the probability  $\mathcal{P}(\phi)$  on  $\Sigma_c$ .
- (iv) By integrating the classical equation of motion from  $\Sigma_c$ , we evaluate the number of e-foldings for each classical trajectories. Then calculate the probability measure of the e-foldings.
- (v) Repeating step (ii) to step (iv) for different values of parameters  $(a, b)$ , we obtain the probability of parameters  $(a, b)$  which specify boundary conditions of the wave function. We calculate the probability for  $9 \times 9$  grid points in the parameter space  $(a, b)$  of boundary wave functions.



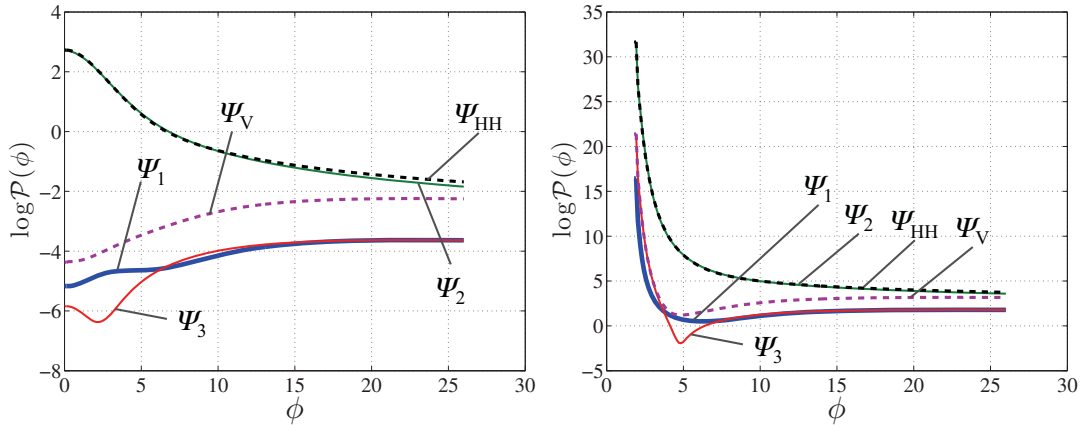
**Figure 1.** The density plot of wave functions with the no-boundary boundary condition (HH). For this boundary condition, wave functions are real. Left:  $\mu = 0.2$ ,  $q_{\text{ini}} = 0.01$ . Right:  $\mu = 3$ ,  $q_{\text{ini}} = 0.0001$ . The dashed line represents  $2qV(\phi) = 1$  which is the boundary between the Euclidean region and the Lorentzian region in mini-superspace.





**Figure 2.** The imaginary part of the wave function with the tunneling boundary condition (V). Left:  $\mu = 0.2$ ,  $q_{\text{ini}} = 0.01$ , Right:  $\mu = 3$ ,  $q_{\text{ini}} = 0.0001$ . The dashed line represents the boundary between the Euclidean region and the Lorentzian region.

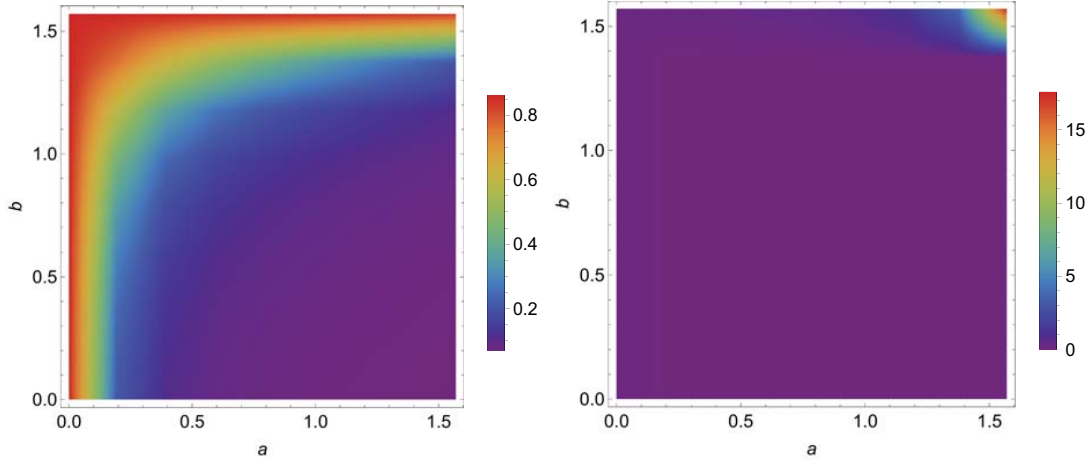
Fig. 1 shows wave functions with the boundary wave function  $\Psi_{\text{HH}}$  (the no-boundary condition (HH)). Fig. 2 shows wave functions with the boundary wave function  $\Psi_{\text{V}}$  (the tunneling boundary condition (V)). Fig. 3 shows  $\mathcal{P}(\phi)$  obtained from solutions of the WD equation with boundary wave functions  $\Psi_1$ ,  $\Psi_2$ ,  $\Psi_3$  and  $\Psi_{\text{HH}}$ ,  $\Psi_{\text{V}}$ . From now on, we denote wave functions with these boundary wave functions as  $\Psi_1, \Psi_2, \Psi_3, \Psi_{\text{HH}}, \Psi_{\text{V}}$ . This probability measure is not normalized because the conditional probability  $P(S|a, b)$  can be obtained without normalizing  $\mathcal{P}(\phi)$ .



**Figure 3.**  $\mathcal{P}(\phi)$  for  $\mu = 0.2$  (left) and  $\mu = 3$  (right).  $\mathcal{P}(\phi)$  is not normalized.

For  $\mu = 0.2$ , wave functions  $\Psi_1$ ,  $\Psi_3$ ,  $\Psi_{\text{V}}$  prefer large values of  $\phi$  while  $\Psi_2$ ,  $\Psi_{\text{HH}}$  prefer small values of  $\phi$ . This behavior of  $\mathcal{P}(\phi)$  is the same as that obtained from the exact wave function  $\Psi_{\text{C}}$ . However, the distribution for small  $\phi$  is slightly different from that obtained by  $\Psi_{\text{C}}$ . For  $\mu = 3$ , probabilities for  $\Psi_2$  and  $\Psi_{\text{HH}}$  have the same behavior as these in the  $\mu = 0.2$  model. However, probabilities for  $\Psi_1$ ,  $\Psi_3$  and  $\Psi_{\text{V}}$  show different behavior; The probabilities for small  $\phi$  have significantly large values for the  $\mu = 3$  model.

Using  $\mathcal{P}(\phi)$ , we obtain the probability measure for number of e-foldings  $\mathcal{N}$  by numerical integrations of classical trajectories starting from  $\Sigma_c$ . Then, we evaluate the conditional probability for the sufficient inflation  $P_{\text{suf}} = P(\mathcal{N} \geq 60)$ . Finally, we obtain the probability  $P(a, b)$  of boundary conditions using the relation (8) (Fig. 4). For  $\mu = 0.2$ ,  $P(a, b)$  has large



**Figure 4.** Density plot of  $P(a, b)$  for  $\mu = 0.2$  (left) and  $\mu = 3$  (right). We evaluated this probability on  $9 \times 9$  grid points in the parameter space of boundary conditions.

values on lines  $a = 0$  and  $b = \pi/2$ . These two lines correspond to the wave functions  $\Psi_1$ ,  $\Psi_3$  and  $\Psi_V$ .  $P(a, b)$  has small values on the line  $b = 0$ , corresponding to  $\Psi_2$  and  $\Psi_{\text{HH}}$ . Consequently,  $\Psi_V$  is more preferable than  $\Psi_{\text{HH}}$  to realize large e-foldings, and this result is the same as one predicted by the wave functions for a constant scalar field potential. In contrast to that, for  $\mu = 3$ ,  $P(a, b)$  has large values only around the point  $(a, b) = (\pi/2, \pi/2)$ , which corresponds to the boundary wave function  $\Psi_1$ . This behavior is significantly different from the  $\mu = 0.2$  case. Superiority of  $\Psi_V$  to  $\Psi_{\text{HH}}$  holds both for  $\mu = 0.2$  and  $\mu = 3$  models. We expect our results with parameter  $\mu = 0.2$  and  $\mu = 3$  are typical ones and  $P(a, b)$  with  $\mu < \mu_*$  behaves similar to the  $\mu = 0.2$  case and  $P(a, b)$  with  $\mu > \mu_*$  behaves similar to the  $\mu = 3$  case.

## 5. Conclusion

In this article, we considered boundary conditions for the wave function of the universe which lead to sufficient e-foldings of inflation. For this purpose, we adopted the exact solutions of the WD equation with a constant scalar field potential as the boundary condition of the wave function, and solved the WD equation numerically. This boundary condition is parametrized with two real parameters and includes both the tunneling and the no-boundary boundary conditions. We obtained the probability distribution function for these parameters under the condition of sufficient e-foldings of inflation. The parameters with large value of this probability determines the boundary condition of the wave function which predicts sufficient e-foldings of inflation. We found that the probability distribution of boundary conditions has two different behavior depending on the value of model parameter  $\mu$ .

For small values of  $\mu$ , the cosmological constant dominates and the inflaton field asymptotically approaches to zero without oscillation. In this case,  $\phi$  dependence of the wave function is not so strong and the obtained wave function reproduces behavior of exact wave functions with a constant scalar field potential. Hence the probability of boundary conditions has large values for  $\Psi_1, \Psi_3, \Psi_V$  and small values for  $\Psi_2, \Psi_{\text{HH}}$ . Thus, boundary conditions

$\Psi_1, \Psi_3, \Psi_V$  are preferable to realize sufficient period of inflation and superiority among them is small. This behavior of the probability of boundary conditions can be expected from behavior of wave functions for a constant scalar field potential. On the other hand, for large values of  $\mu$ , the slow roll inflation is followed by oscillation of the inflaton field. In this case, the derivative term of  $\phi$  in the WD equation cannot be neglected and wave functions have large values about  $\phi = 0$  for any boundary conditions. Owing to this behavior of the wave function, the probability of boundary conditions has large value about  $\Psi_1$  (superiority of  $\Psi_V$  over  $\Psi_{HH}$  is kept as before). Thus, realistic inflationary models followed by oscillation of inflaton field select the boundary condition  $\Psi_1$ .

As an extension of analysis presented in this article, it is also possible to discuss probability for values of the model parameter  $\mu$ . The probability distribution of boundary conditions has a sharp peak for the model with large  $\mu$  and selects a specific boundary condition. This implies that a suitable boundary condition is automatically chosen for large values of  $\mu$  (small values of the cosmological constant). If we assume that the probability of boundary conditions select an unique boundary condition, the parameter  $\mu$  must acquire large value (the cosmological constant must be small).

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