

# Extensions of Born's rule to non-linear quantum mechanics, some of which do not imply superluminal communication

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**Abstract.** Nonlinear modifications of quantum mechanics have a troubled history. They were initially studied for many promising reasons: resolving the measurement problem, formulating a theory of quantum mechanics and gravity, and understanding the limits of standard quantum mechanics. However, certain non-linear theories have been experimentally tested and failed. More significantly, it has been shown that, in general, deterministic non-linear theories can be used for superluminal communication. We highlight another serious issue: the distribution of measurement results predicted by non-linear quantum mechanics depends on the formulation of quantum mechanics. In other words, Born's rule cannot be uniquely extended to non-linear quantum mechanics. We present these generalizations of Born's rule, and then examine whether some exclude superluminal communication. We determine that a large class do not allow for superluminal communication, but many lack a consistent definition. Nonetheless, we find a single extension of Born's rule with a sound operational definition, and that does not exhibit superluminal communication. The non-linear time-evolution leading to a certain measurement event is driven by the state conditioned on measurements that lie within the past light cone of that event.

## 1. Introduction

Non-linear quantum mechanics (NLQM) has long been considered as a possible generalization of standard quantum mechanics (sQM) [1, 2, 3], for three main reasons. First, the measurement process is controversial, because if we assume that linear quantum mechanics explains all processes, then it is very difficult to explain wavefunction collapse [4]. Phenomenological non-linear stochastic, and experimentally falsifiable, extensions of quantum mechanics have been proposed to explain the measurement process [5], and upper bounds on the parameters of such theories have been obtained in [5, 6, 7]. Second, we would like to test the domain of validity of sQM. One possible feature to test is linearity. Experimental tests of certain non-linear theories have been performed in [8, 9, 10, 11, 12], all with negative results. Third, non-linear and deterministic theories of quantum mechanics have been proposed to unite quantum mechanics with gravity. For instance, the Schroedinger-Newton theory describes a classical spacetime which is sourced by quantum matter [13], and the correlated worldlines theory is a quantum theory of gravity, which postulates that gravity correlates quantum trajectories [14].

Despite these appealing reasons to study NLQM, there was a backlash after Gisin showed that deterministic NLQM could imply superluminal communication [15]. The no-signaling condition



states that one cannot send information faster than the speed of light, and is a cornerstone of the special theory of relativity. The community regards the condition as being inviolable. Gisin's work was quickly followed by others with similar conclusions [16, 17]. Additional work then showed that under general conditions NLQM implies superluminal communication [18, 19]. In this article, we show that one can still satisfy the no-signaling condition by exploiting an ambiguity in Born's rule when it is extended to NLQM.

Along with the Schroedinger equation, Born's rule is a fundamental tenet of quantum mechanics. The former determines the time evolution of a wavefunction, while Born's rule provides the probabilistic interpretation of the wavefunction, which allows one to make predictions about the distribution of measurement results. Born's rule has, so far, passed all experimental tests, and so any non-linear theory must make predictions that become equivalent to Born's rule in sQM when the non-linearity vanishes.

In [6], we showed that NLQM suffers from a serious conceptual issue: Born's rule cannot be uniquely extended to NLQM. Specifically, there are multiple prescriptions of assigning the probability of a measurement outcome, that are equivalent in sQM, but become distinct in NLQM. Note that we treat Born's rule as a phenomenological prescription for determining the distribution of measurement results, and so there is no *a priori* reason to pick one extension over another.

We first present an ambiguity in the choice of the state we non-linearly evolve, by providing a simple example involving a single measurement. We then discuss a more elaborate setup, which is similar to that used to show that non-linear theories violate the no-signaling condition. We determine that there are no prescriptions based on this ambiguity that do not result in superluminal communication. In section 3, we present another type of ambiguity: the choice of wavefunction that drives the non-linear time evolution. We determine that there is an infinite number of prescriptions based on this ambiguity that do not violate the no-signaling condition. However, the majority do not have a sensible interpretation. In section 5, we present and discuss a particular prescription with a reasonable operational definition and that does not exhibit superluminal communication.

## 2. Ambiguity in choosing which state is non-linearly time evolved

In this section, we discuss one type of ambiguity that we encounter when extending Born's rule to NLQM: a choice in the state that we time-evolve. We illustrate this with a simple example, and then discuss whether the generated class exhibits superluminal signaling.

### 2.1. A simple example

According to Born's rule, if a system is initially in a state  $|i\rangle$ , the probability that a measurement device measures it to be in some eigenstate  $|f\rangle$ , after some period of evolution under a unitary operator  $\hat{U}$ , is

$$p_{i \rightarrow f} = \left| \langle f | \hat{U} | i \rangle \right|^2. \quad (1)$$

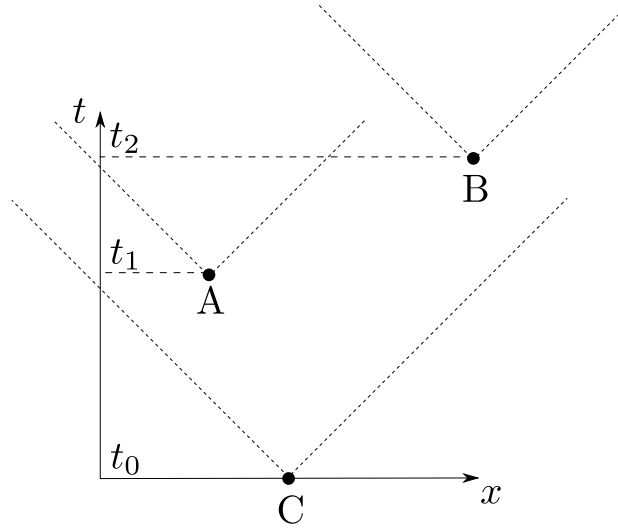
The above formula can be written in a number of equivalent ways, such as

$$p_{i \leftarrow f} = \left| \langle i | \hat{U}^\dagger | f \rangle \right|^2. \quad (2)$$

In NLQM, the time evolution operator depends on the state it acts on. As a result, Eqs. (1) and (2) become

$$p_{i \rightarrow f}^{\text{NL}} = |\langle f | \mathcal{U}_i | i \rangle|^2, \quad p_{i \leftarrow f}^{\text{NL}} \propto \left| \langle i | \mathcal{U}_f^\dagger | f \rangle \right|^2, \quad (3)$$

where  $|\mathcal{U}_i i\rangle$  is the time-evolved  $|i\rangle$  and  $|\mathcal{U}_f^\dagger f\rangle$  is the backwards time-evolved  $|f\rangle$ , under some non-linear dynamics. The superscript NL explicitly indicates that NLQM is being used. Moreover,



**Figure 1.** A setup with multiple measurement events. Event C describes the preparation of an ensemble of identical 2-particle states  $|\Psi_{\text{ini}}\rangle$  by Charlie. Event A (B) describes Alice (Bob) measuring her (his) particles. The dashed lines show the light cone centered around each event.

the proportionality sign in  $p_{i \leftarrow f}^{\text{NL}}$  follows from  $\sum_f \left| \langle i | \mathcal{U}_f^\dagger f \rangle \right|^2$  being not, in general, normalized to unity.  $p_{i \rightarrow f}^{\text{NL}}$  and  $p_{i \leftarrow f}^{\text{NL}}$  are not necessarily equivalent, and so Born's rule cannot be uniquely extended to NLQM.

## 2.2. Multiple measurements

Having established the ambiguity of Born's rule, we move on to study more complicated setups with multiple measurements. Our aim is determining whether there exists an extension of Born's rule that does not exhibit superluminal communication, and so we will study the prototypical setup for showing that NLQM violates the no-signaling condition. As shown in Fig. 1, Charlie prepares a collection of identical *arbitrary* 2-particle states  $|\Psi_{\text{ini}}\rangle$ , and then sends them to Alice and Bob, such that they both hold one part of each of the states  $|\Psi_{\text{ini}}\rangle$ . Alice performs measurements on her ensemble of particles at  $t_1$ , and then Bob on his at a later time  $t_2$ . We assume that Alice's measurement events are space-like separated from Bob's. The ordering of  $t_1$  and  $t_2$ , which depends on the reference frame that the experiment is viewed in, won't affect our analysis.

Denote the probability that Alice measures  $\alpha$  in some basis  $z$ , and Bob measures  $\beta$  in some basis  $x$  by  $p(\alpha, \beta)$ . For example, if the particles were spins,  $z$  could be the  $\hat{\sigma}_z$  eigenstates,  $|\uparrow\rangle, |\downarrow\rangle$ , and  $x$  could be the  $\hat{\sigma}_x$  eigenstates  $|\pm\rangle = (|\uparrow\rangle \pm |\downarrow\rangle) / \sqrt{2}$ . We will first determine the probability distribution of different measurement events according to sQM, and then discuss the different ways of generalizing the probability distribution to NLQM.

In sQM,  $p(\alpha, \beta)$  is given by

$$p(\alpha, \beta) = \langle \Psi_{f|\alpha, \beta} | \Psi_{f|\alpha, \beta} \rangle, \quad (4)$$

where  $|\Psi_{f|\alpha, \beta}\rangle$  is the unnormalized joint quantum state of Alice and Bob at time  $t_2$ , conditioned on the measurement results  $\alpha$  and  $\beta$ :

$$|\Psi_{f|\alpha, \beta}\rangle = (\hat{I} \otimes \hat{P}_\beta) \hat{U}_2 (\hat{P}_\alpha \otimes \hat{I}) \hat{U}_1 |\Psi_{\text{ini}}\rangle, \quad (5)$$

where  $\hat{U}_1$  is the total time evolution operator (for both Alice and Bob's particles) from the initial time  $t_0$  till  $t_1$ , and  $\hat{U}_2$  is the total time evolution operator from  $t_1$  till  $t_2$ . The projectors are  $\hat{P}_\alpha = |\alpha\rangle\langle\alpha|$  and  $\hat{P}_\beta = |\beta\rangle\langle\beta|$ . Note that we chose to work with pure states instead of density matrices for clarity and simplicity.  $|\Psi_{\text{ini}}\rangle$  can always be enlarged to include the initial state of the environment.

Substituting the expression of  $|\Psi_{f|\alpha,\beta}\rangle$  into Eq. (4), we have

$$p(\alpha, \beta) = \left\langle \Psi_{\text{ini}} \left| \hat{U}_1^\dagger \hat{P}_\alpha \hat{U}_2^\dagger \hat{P}_\beta \hat{U}_2 \hat{P}_\alpha \hat{U}_1 \right| \Psi_{\text{ini}} \right\rangle. \quad (6)$$

$p(\alpha, \beta)$  can also be written in a more convenient form by writing the projection operators in the Heisenberg picture:

$$\hat{P}_\alpha^H \equiv \hat{U}_1^\dagger \hat{P}_\alpha \hat{U}_1; \quad \hat{P}_\beta^H \equiv \hat{U}_{\text{tot}}^\dagger \hat{P}_\beta \hat{U}_{\text{tot}}, \quad \hat{U}_{\text{tot}} \equiv \hat{U}_2 \hat{U}_1. \quad (7)$$

We rewrite  $p(\alpha, \beta)$  in terms of  $\hat{P}_\alpha^H$  and  $\hat{P}_\beta^H$ :

$$p(\alpha, \beta) = \left\langle \Psi_{\text{ini}} \left| \hat{P}_\beta^H \hat{P}_\alpha^H \right| \Psi_{\text{ini}} \right\rangle, \quad (8)$$

where we have used that Alice and Bob's measurement events are space-like separated, and so their corresponding measurement operators commute. This concise way of writing  $p(\alpha, \beta)$  has the advantage of being easily generalizable to more than just 2 measurements.

$p(\alpha, \beta)$  can be extended to uncountably many ways in NLQM. For instance, we can exploit the fact that for any projection operator  $\hat{P}^\dagger \hat{P} = \hat{P}$ , and that  $\hat{I}$  can be decomposed as the product of any unitary operator and its hermitian conjugate (e.g.  $\hat{I} = \hat{U}_1^\dagger \hat{U}_1$ ). Each unitary operator, when extended to NLQM, can act on the state to its left or to its right. Note that, in general, we will need to add a normalization factor to  $p^{\text{NL}}(\alpha, \beta)$ :

$$p^{\text{NL}}(\alpha, \beta) = \frac{1}{\mathcal{N}} \left\langle \Psi_{f|\alpha,\beta} | \Psi_{f|\alpha,\beta} \right\rangle, \quad (9)$$

where

$$\mathcal{N} \equiv \sum_{\gamma,\delta} \left\langle \Psi_{f|\gamma,\delta} | \Psi_{f|\gamma,\delta} \right\rangle. \quad (10)$$

Can we exploit this ambiguity to find a prescription that does not violate the no-signaling communication?

### 2.3. Imposing the no-signaling condition

Superluminal communication from Alice to Bob is not possible when

$$p(\beta) = \sum_{\alpha} p(\alpha, \beta) \quad (11)$$

is independent of the basis chosen by Alice. Otherwise, if  $p(\beta)$  is influenced by Alice's choice of measurement basis in a deterministic way, then since Bob can easily estimate  $p(\beta)$ , he can determine the basis Alice chose for her measurement results, which can form the foundation of a communication strategy. For instance, both Alice and Bob can agree that a particular choice of Alice's measurement basis could be associated with sending the bit 0, while another choice could be associated with the bit 1.

In other words, to satisfy the no-signaling condition, we require that  $p(\beta)$  depends only on  $\beta$ . In sQM, this is clearly satisfied, as

$$p(\beta) = \left\langle \Psi_{\text{ini}} \left| \hat{P}_\beta^H \left( \sum_\alpha \hat{P}_\alpha^H \right) \right| \Psi_{\text{ini}} \right\rangle = \left\langle \Psi_{\text{ini}} \left| \hat{P}_\beta^H \right| \Psi_{\text{ini}} \right\rangle, \quad (12)$$

because

$$\sum_\alpha \hat{P}_\alpha^H = \hat{U}_1^\dagger \left( \sum_\alpha |\alpha\rangle \langle \alpha| \right) \hat{U}_1 = \hat{I}. \quad (13)$$

We now analyze superluminal communication for general non-linear theories. We would like  $\sum_\alpha p^{\text{NL}}(\alpha, \beta)$  to be independent of  $\alpha$ , so  $\sum_\alpha \langle \Psi_{f|\alpha, \beta} | \Psi_{f|\alpha, \beta} \rangle$  has to be independent of  $\alpha$  for all  $\beta$ . Note that the normalization cannot, in general, help. A non-trivial measurement apparatus has at least two outcomes, so both the numerator and denominators of  $p^{\text{NL}}(\alpha, \beta)$  consist of a sum of at least two terms. The only way that  $\mathcal{N}$  gets rid of the dependence of each term in the numerator on the different  $\alpha$ s is if each term is of the form  $A(\{\alpha\}) B(\beta)$ , where  $B(\beta)$  is a general function that depends only  $\beta$ , and  $A(\{\alpha\})$  is a function identical among all the  $\langle \Psi_{f|\alpha, \beta} | \Psi_{f|\alpha, \beta} \rangle$ . Such a separable expression is not in general possible, because the initial state can be chosen to be anything, including entangled states, and the non-linear dynamics are arbitrary.

We can choose  $p^{\text{NL}}(\beta)$  to be independent of the different  $\alpha$ s. This can be seen by first expanding Eq. (8):

$$p(\alpha, \beta) = \left\langle \Psi_{\text{ini}} \left| \hat{U}_{\text{tot}}^\dagger \hat{P}_\beta \hat{U}_2 \hat{P}_\alpha \hat{U}_1 \right| \Psi_{\text{ini}} \right\rangle. \quad (14)$$

When extending  $p(\alpha, \beta)$  to  $p^{\text{NL}}(\alpha, \beta)$ , let  $\hat{U}_{\text{tot}}^\dagger$  act on  $\langle \Psi_{\text{ini}} |$ :

$$\langle \Psi_{\text{ini}} | \hat{U}_{\text{tot}}^\dagger \rightarrow \langle \mathcal{U}_{\text{tot}} \Psi_{\text{ini}} |. \quad (15)$$

Let  $\hat{U}_2$  act the conditional state  $\langle \mathcal{U}_{\text{tot}} \Psi_{\text{ini}} | \hat{P}_\beta$ , which we will define as  $\Psi_c$ :

$$\hat{P}_\beta |\mathcal{U}_{\text{tot}} \Psi_{\text{ini}}\rangle \equiv |\Psi_c\rangle \quad (16)$$

$$\langle \Psi_{\text{ini}} | \hat{U}_{\text{tot}}^\dagger \hat{P}_\beta \hat{U}_2 \rightarrow \langle \mathcal{U}_2^\dagger \Psi_c |, \quad (17)$$

where  $\mathcal{U}_2^\dagger$  denotes backwards time evolution. Finally, let  $\hat{U}_1$  act on  $|\Psi_{\text{ini}}\rangle$ :

$$p^{\text{NL}}(\alpha, \beta) = \frac{1}{\mathcal{N}} \left\langle \mathcal{U}_2 \Psi_c \left| \hat{P}_\alpha \right| \mathcal{U}_1 \Psi_{\text{ini}} \right\rangle. \quad (18)$$

Notice that

$$p^{\text{NL}}(\beta) = \sum_\alpha p^{\text{NL}}(\alpha, \beta) = \frac{1}{\mathcal{N}} \left\langle \mathcal{U}_2 \Psi_c \left| \left( \sum_\alpha \hat{P}_\alpha \right) \right| \mathcal{U}_1 \Psi_{\text{ini}} \right\rangle \quad (19)$$

depends only on  $\beta$  because the sum over  $\alpha$  can be brought over to inside the expectation value, where it acts solely on  $\hat{P}_\alpha$  and gives  $\hat{I}$ .

We have ensured that Alice cannot communicate with Bob faster than the speed of light. However, Bob can superluminally communicate with Alice, because when Eq. (18) is summed over  $\beta$ , the sum cannot be brought over to  $\hat{P}_\beta$ , and so in general  $\sum_\beta p^{\text{NL}}(\alpha, \beta)$  does not depend only on  $\alpha$ . There is no way to prevent superluminal communication with the analysis performed above, as was concluded in [18]. Nonetheless, we have shown that some prescriptions can prevent one party from communicating to another.

In the next section, we exploit one more ambiguity to show that some extensions of Born's rule do not violate the no-signaling condition.

### 3. Ambiguity in the state driving the non-linear time evolution

We can take advantage of an unorthodox ambiguity to meet the no-signaling condition. We first present the ambiguity without regards to interpretation, and then investigate whether extensions of Born's rule that exploit this ambiguity have a reasonable interpretation, and a consistent operational definition.

#### 3.1. The ambiguity

In general, the non-linear Schroedinger equation contains a nonlinear term  $\hat{V}_{\text{NL}}$  and a linear term,  $\hat{H}_L$ :

$$i\hbar\partial_t |\psi\rangle = \left( \hat{H}_L + \hat{V}_{\text{NL}}(\psi) \right) |\psi\rangle, \quad (20)$$

where  $\hat{V}_{\text{NL}}(\psi)$  is a shorthand for a non-linear potential that can depend arbitrarily on  $\psi(z)$  for all times  $z$ . When Eq. (20) is used to predict distributions of measurement outcomes,  $\hat{V}_{\text{NL}}$  does not have to act on  $\psi$ . Instead, it can act on a pre-determined wavefunction  $\phi$ :

$$i\hbar\partial_t |\psi\rangle = \left( \hat{H}_L + \hat{V}_{\text{NL}}(\phi) \right) |\psi\rangle. \quad (21)$$

Notice that, unless  $\phi$  is chosen to be  $\psi$ , Eq. (21) is a *linear* evolution equation, and associated to it is a unitary time evolution operator which we define as  $\hat{U}(\phi)$ . In the limit that  $\hat{V}_{\text{NL}}$  vanishes,  $\hat{U}(\phi)$  recovers the time evolution operator predicted by sQM. Consequently, in the same limit, extensions of Born's rule based on Eq. (21) recover the Born's rule in sQM.

#### 3.2. Imposing the no-signaling condition

Consider the following extension of Eq. (8):

$$p^{\text{NL}}(\alpha, \beta) = \left\langle \Psi_{\text{ini}} \left| \hat{P}_{\beta}^{H_{\phi}} \hat{P}_{\alpha}^{H_{\phi}} \right| \Psi_{\text{ini}} \right\rangle, \quad (22)$$

where the superscript  $H_{\phi}$  denotes a Heisenberg picture for the linear Schroedinger equation given by Eq. (21). For instance,  $\hat{P}_{\alpha}^H$  becomes

$$\hat{P}_{\alpha}^{H_{\phi}} \equiv \hat{U}_1^{\dagger}(\phi) \hat{P}_{\alpha} \hat{U}_1(\phi). \quad (23)$$

As long as  $\phi$  does not depend on  $\alpha$  or  $\beta$ , the expression for  $p^{\text{NL}}(\alpha, \beta)$  given by Eq. (22) does not violate the no-signaling condition.

#### 3.3. Interpretation

Eq. (22) describes extensions of Born's rule that do not exhibit superluminal communication, but most do not have a reasonable operational definition. The issue that we immediately encounter is the choice of  $\phi$  in Eq. (22). In [6], we gave two reasonable choices for  $\phi$ . The first, which we termed *pre-selection*, takes  $\phi$  to be the initial state of the experiment  $|\Psi_{\text{ini}}\rangle$  and its time evolution under the full non-linear Schroedinger equation given by Eq. (20). The second, which we termed *post-selection*, takes  $\phi$  to be the final state of the experiment  $|\alpha, \beta\rangle$  and its (backwards) time evolution under Eq. (20).

These prescriptions have issues. Post-selection violates the no-signaling condition because all evolution operators in Eq. (22) depend on the measurement outcomes  $\alpha$  and  $\beta$ . Pre-selection does not allow superluminal communication. However, choosing  $|\Psi_{\text{ini}}\rangle$  is a delicate matter. Consider again the setup shown by Fig. 1. We chose that  $|\Psi_{\text{ini}}\rangle$  be prepared by a third party: Charlie. In the analysis of the canonical EPR setup shown by Fig. 1, such a statement doesn't affect the calculation of the distribution of measurement results, but in pre-selection it does.

Charlie must have manipulated some state, which we call  $|\Psi'_{\text{ini}}\rangle$ , to prepare  $|\Psi_{\text{ini}}\rangle$ . Consequently, if pre-selection were to be applied, then it should be on  $|\Psi'_{\text{ini}}\rangle$  instead of  $|\Psi_{\text{ini}}\rangle$ . This process could, in principle, be repeated *ad infinitum* back to the initial state of the universe, which is unknown to us. Even if it were known, working with the initial state of the universe would make the analysis of any experiment infeasible.

#### 4. A sensible extension of Born's rule

In this section, we propose a sensible extension of Born's rule, where the state that drives the non-linear time evolution up to a certain measurement event is chosen to be the time evolved initial state conditioned on measurement results that lie within or on the past light cone. Our scheme is similar to Adrian Kent's proposal in [20]. He argued that if the non-linear evolution depends only on local states, defined by allowing only projective measurements in the past light cone of a particle and then tracing out the rest of the degrees of freedom, then superluminal communication is not possible.

##### 4.1. A general example

To clarify and justify our prescription, we present a general example. Consider the setup shown in Fig. 2, which is a more elaborate version of Fig. 1. The thought experiment now includes two additional parties: Dylan, who performs a measurement in the past light cone of Charlie, and Eve who performs a measurement in the future light cone of Bob, and outside the future light cone of Alice. Dylan is included to demonstrate that our proposed prescription does not suffer from the same drawback as pre-selection, and Eve is included to show that even for a complicated configuration of measurement events, our prescription does not exhibit superluminal communication.

We begin our analysis with the predictions of sQM for the final state of the experiment conditioned on Charlie, Alice, Bob and Eve measuring  $C$ ,  $\alpha$ ,  $\beta$  and  $\epsilon$  with corresponding measurement eigenstates  $|\Psi_{\text{ini}}\rangle$ ,  $|\alpha\rangle$ ,  $|\beta\rangle$  and  $|\epsilon\rangle$ , respectively. This conditional state is given by

$$|\psi_c\rangle \propto \hat{P}_\epsilon \hat{U}_3 \hat{P}_\beta \hat{U}_2 \hat{P}_\alpha \hat{U}_1 \hat{P}_C \hat{U}_0 |\Psi'_{\text{ini}}\rangle \propto \hat{P}_\epsilon \hat{U}_3 \hat{P}_\beta \hat{U}_2 \hat{P}_\alpha \hat{U}_1 |\Psi_{\text{ini}}\rangle, \quad (24)$$

where  $|\Psi'_{\text{ini}}\rangle$  is the initial state prepared by Dylan,  $\hat{U}_0$  denotes time evolution from the time of Dylan's measurement till  $t_0$ ,  $\hat{U}_1$  from  $t_0$  till  $t_1$ ,  $\hat{U}_2$  from  $t_1$  till  $t_2$  and  $\hat{U}_3$  from  $t_2$  till  $t_3$ . The projection operators are defined by

$$\hat{P}_C = |\Psi_{\text{ini}}\rangle \langle \Psi_{\text{ini}}|, \quad \hat{P}_\alpha = |\alpha\rangle \langle \alpha|, \quad \hat{P}_\beta = |\beta\rangle \langle \beta|, \quad \hat{P}_\epsilon = |\epsilon\rangle \langle \epsilon|. \quad (25)$$

According to our proposed prescription,  $|\psi_c\rangle$  extends to NLQM in the following way:

$$|\psi_c\rangle \propto \hat{P}_\epsilon \hat{U}_3 \left( \hat{P}_\beta \Psi_{\text{ini}} \right) \hat{P}_\beta \hat{U}_2 (\Psi_{\text{ini}}) \hat{P}_\alpha \hat{U}_1 (\Psi_{\text{ini}}) \hat{P}_C \hat{U}_0 \left( \Psi'_{\text{ini}} \right) |\Psi'_{\text{ini}}\rangle \quad (26)$$

$$\propto \hat{P}_\epsilon \hat{U}_3 \left( \hat{P}_\beta \Psi_{\text{ini}} \right) \hat{P}_\beta \hat{U}_2 (\Psi_{\text{ini}}) \hat{P}_\alpha \hat{U}_1 (\Psi_{\text{ini}}) |\Psi_{\text{ini}}\rangle, \quad (27)$$

where we remind the reader that  $\hat{U}(\phi)$  is the time evolution operator associated with Eq. (21). Charlie's measurement event lies in the past light cone of Alice, so time evolution up to Alice's measurement is driven by the conditional state  $\hat{P}_C \hat{U}_0 \left( \Psi'_{\text{ini}} \right) |\Psi'_{\text{ini}}\rangle$  which is just  $|\Psi_{\text{ini}}\rangle$ . Bob's past light cone does not include Alice's measurement event, but includes Charlie's, so we have to use the conditional state at Charlie,  $|\Psi_{\text{ini}}\rangle$ , to drive the non-linear time evolution up to Bob's



measurement. Finally, Eve's past light cone includes Bob's measurement event, so we use the conditional state  $\hat{P}_\beta \hat{U}_1(\Psi_{\text{ini}}) |\Psi_{\text{ini}}\rangle$  to drive the nonlinear time evolution between Bob and Eve.

Eq. (27) contains genuine non-linear time evolution, such as  $\hat{U}_0(\Psi'_{\text{ini}}) |\Psi'_{\text{ini}}\rangle$ . This equation predicts dynamics that are identical to those predicted by the full non-linear Schroedinger equation given by Eq. (20) with an initial state of  $|\Psi'_{\text{ini}}\rangle$ . Moreover, notice that measurements within the past light cone of Charlie, like that of Dylan's, do not affect our analysis. Indeed, preparation events are always in the past light cone of the final measurements of an experiment, because the particles, whose speed is upper bounded by the speed of light, have to travel to where they will be measured. Consequently, our proposed scheme avoids the issue of pre-selection, which is the difficulty of choosing the initial state that will drive the non-linear time evolution.

We show that our proposed prescription does not violate the no-signaling condition by looking at the marginal probabilities,  $p(\alpha|C)$ ,  $p(\beta|C)$  and  $p(\epsilon|C)$ , conditioned on Charlie measuring  $|\Psi_{\text{ini}}\rangle$ . The probability of obtaining the measurement results  $\alpha, \beta, \epsilon$ , and that Charlie measures  $|\Psi_{\text{ini}}\rangle$  is given by the norm of  $|\Psi_c\rangle$  in Eq. (27):

$$\begin{aligned} & p^{\text{NL}}(\alpha, \beta, \epsilon, C) \\ &= \left| \langle \Psi_{\text{ini}} | \hat{U}_0(\Psi'_{\text{ini}}) |\Psi'_{\text{ini}}\rangle \right|^2 / \mathcal{N} \times \\ & \quad \langle \Psi_{\text{ini}} | \hat{U}_1^\dagger(\Psi_{\text{ini}}) \hat{U}_2^\dagger(\Psi_{\text{ini}}) \hat{P}_\beta \hat{U}_3^\dagger(\hat{P}_\beta \Psi_{\text{ini}}) \hat{P}_\epsilon \hat{U}_3(\hat{P}_\beta \Psi_{\text{ini}}) \hat{P}_\beta \hat{U}_2(\Psi_{\text{ini}}) \hat{P}_\alpha \hat{U}_1(\Psi_{\text{ini}}) |\Psi_{\text{ini}}\rangle, \end{aligned}$$

where  $\mathcal{N}$  is a normalization factor given by the sum of  $p^{\text{NL}}(\alpha, \beta, \epsilon, C)$  over all possible measurement results. We calculate it to be unity. We have also simplified  $p^{\text{NL}}(\alpha, \beta, \epsilon, C)$  by noting that  $\hat{P}_\alpha$ , which corresponds to an event space-like separated from Bob and Eve's measurements, commutes with  $\hat{U}_2$ ,  $\hat{U}_3$ ,  $\hat{P}_\beta$  and  $\hat{P}_\epsilon$ . We are interested in  $p^{\text{NL}}(\alpha, \beta, \epsilon|C)$ , so we have to divide by  $p^{\text{NL}}(C)$ , which can be obtained by summing  $p^{\text{NL}}(\alpha, \beta, \epsilon, C)$  over  $\alpha, \beta$  and  $\epsilon$ :

$$\begin{aligned} & p^{\text{NL}}(\alpha, \beta, \epsilon|C) \\ &= \langle \Psi_{\text{ini}} | \hat{U}_1^\dagger(\Psi_{\text{ini}}) \hat{U}_2^\dagger(\Psi_{\text{ini}}) \hat{P}_\beta \hat{U}_3^\dagger(\hat{P}_\beta \Psi_{\text{ini}}) \hat{P}_\epsilon \hat{U}_3(\hat{P}_\beta \Psi_{\text{ini}}) \hat{P}_\beta \hat{U}_2(\Psi_{\text{ini}}) \hat{P}_\alpha \hat{U}_1(\Psi_{\text{ini}}) |\Psi_{\text{ini}}\rangle. \end{aligned}$$

#### 4.2. Superluminal signaling is forbidden

Can Alice send signals to Bob or Eve, or vice versa? Let us first calculate  $p^{\text{NL}}(\beta|C)$ :

$$p^{\text{NL}}(\beta|C) = \langle \Psi_{\text{ini}} | \hat{U}_1^\dagger(\Psi_{\text{ini}}) \hat{U}_2^\dagger(\Psi_{\text{ini}}) \hat{P}_\beta \hat{U}_2(\Psi_{\text{ini}}) \hat{U}_1(\Psi_{\text{ini}}) |\Psi_{\text{ini}}\rangle, \quad (28)$$

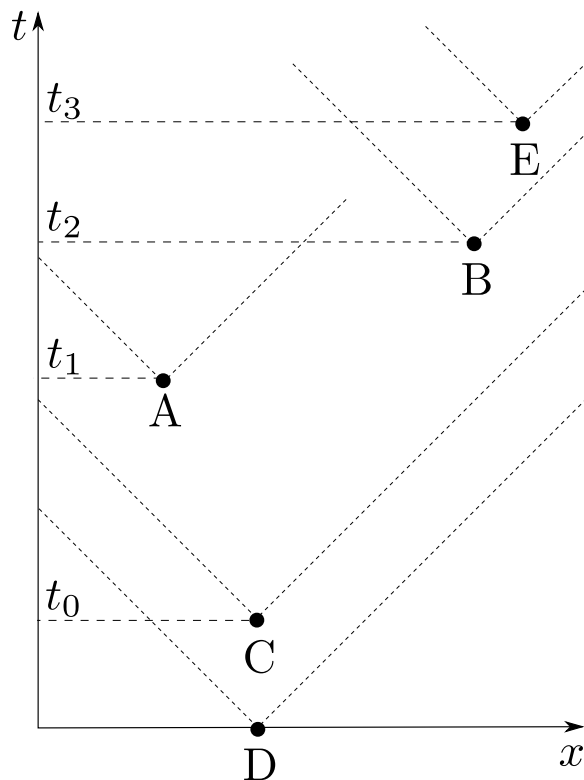
which only depends on  $\beta$ . Next, let us look at Eve's distribution of measurement results. We obtain that  $p^{\text{NL}}(\epsilon|C)$  depends on  $\{\beta\}$  so Bob can send signals to Eve. This is acceptable as the distance between their measurement events is time-like. Finally, Alice cannot violate the no-signaling condition because

$$p^{\text{NL}}(\alpha|C) = \langle \Psi_{\text{ini}} | \hat{U}_1^\dagger(\Psi_{\text{ini}}) \hat{P}_\alpha \hat{U}_1(\Psi_{\text{ini}}) |\Psi_{\text{ini}}\rangle \quad (29)$$

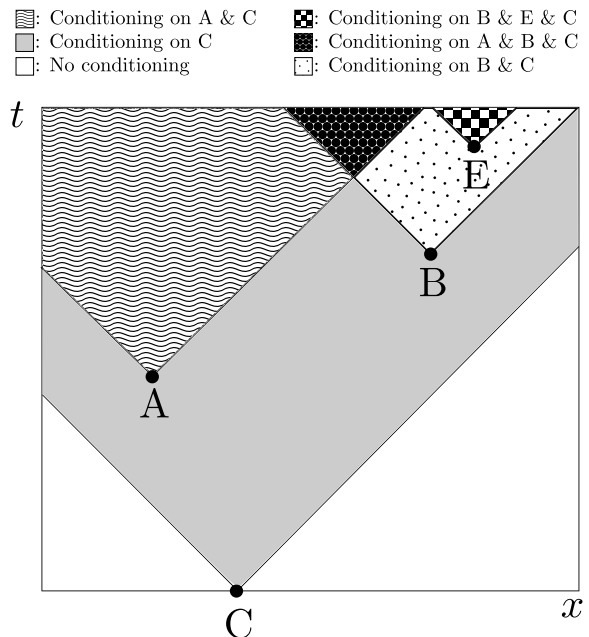
depends only on  $\alpha$ .

The prescription described in this section has genuine non-linear time evolution, and yet does not violate the no-signaling condition. We do not contradict the conclusions of [18], because we violate one of the general conditions they stated are needed for deterministic NLQM to have superluminal communication. In particular, we break the assumption that states are represented by vectors in a Hilbert space. In Eq. (21), 2 states determine time evolution:  $\psi$ , and  $\phi$ , which drives the non-linear time evolution under  $\hat{V}_{\text{NL}}$ . The choice of  $\phi$  to use depends on the spacetime region the observer is in, as is depicted in Fig. 3.





**Figure 2.** A setup similar to that described by Fig. 1, but more elaborate. Event D is Dylan preparing the state  $|\Psi'_{\text{ini}}\rangle$ , Event C is Charlie measuring the eigenstate  $|\Psi_{\text{ini}}\rangle$ . Event A (B) describes Alice (Bob) measuring her (his) particles. The dashed lines show the light cone centered around each event.



**Figure 3.** Partitioning of spacetime into different regions according to which wavefunction drives the non-linear time evolution. There are 4 measurement events: C, A, B and E, arranged identically to the events in Fig. 2. We didn't include Event D to limit clutter. The 4 events result in 6 regions. In each, the wavefunction which drives the non-linear time evolution is the time-evolved initial state of the experiment conditioned on measurements which are presented in the legend at the top of the figure.

## 5. Conclusions

In summary, Born's rule cannot be uniquely extended to non-linear quantum mechanics. This ambiguity seems like another significant penalty for breaking linearity in sQM. However, when we demanded a formulation of NLQM that does not violate the no-signaling condition and that has a reasonable operational definition, we arrived at a single extension of Born's rule. Specifically, we chose that the non-linear time-evolution leading to a certain measurement event is driven by the time-evolved initial state of the experiment conditioned on measurements that lie within the past light cone of that event.

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