Analytical study of fractional equations describing anomalous diffusion of energetic particles

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Abstract. To present the main influence of anomalous diffusion on the energetic particle propagation, the fractional derivative model of transport is developed by deriving the fractional modified Telegraph and Rayleigh equations. Analytical solutions of the fractional modified Telegraph and the fractional Rayleigh equations, which are defined in terms of Caputo fractional derivatives, are obtained by using the Laplace transform and the Mittag-Leffler function method. The solutions of these fractional equations are given in terms of special functions like Fox's H, Mittag-Leffler, Hermite and Hyper-geometric functions. The predicted travelling pulse solutions are discussed in each case for different values of fractional order.

1. Introduction

Anomalous diffusion is an ubiquitous phenomenon and one of the fundamental processes for transport of matter in different systems [1-2]. The anomalous diffusion, based on the mean square displacement of the diffusing species $< z^2(t) \sim t^\alpha >$ for a slow process with $0 < \alpha < 1$ is called sub-diffusive. The sub-diffusion process has been observed in many physical systems such as spatially disordered or fractal media and in the temporal fluctuations of the medium [3]. It is well- known that sub-diffusion equations in terms of fractional derivatives can be obtained from Continuous Time Random Walk models [4]. In general, the anomalous regimes are characterized by non-Gaussian statistics like Levy statistics, which encompasses probability distributions with power-law tails. One of the methods describing the anomalous process is to replace the ordinary time and space derivatives in a standard kinetic equation by fractional time and space derivatives. In this paper, we introduce the fractional Klein-Kramer equation which can be considered as an extension of the fractional diffusion equation by using the Caputo fractional derivative [5]. Caputo's definition the derivative of any continuous function $\phi(t)$ is as follows

$$^cD_t^\alpha\phi(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{d^n}{d\tau^n}\phi(\tau)\,d\tau,\ n-1 < \alpha < n, n \geq 0,\ n \in N \qquad (1)$$

Some endeavors to understand the energetic particle transport in the cosmos [6] consider the Langevin equation [7] for the coordinate $z(t)$

$$\frac{dv}{dt} = -\eta v + \frac{F(z)}{m} + \Gamma(t), \quad \frac{dz}{dt} = v, \qquad (2)$$

where m is the mass of a diffusing particle, η denotes a friction coefficient, $F(z)$ is the external force field and $\Gamma(t)$ is a Gaussian white noise and represent the source of the anomalous behavior. Below we introduce the approximations of the fractional Klein-Kramer equation to describe the sub-diffusion process. Two limiting cases can be distinguished, namely the fractional modified Telegraph from which the probability distribution function can be derived and the fractional Rayleigh equation controlling the velocity distribution in the force-free limit. The paper is organized as follows. In section 2, the fractional modified Telegraph and the fractional Rayleigh equations are derived from the fractional Klein-Kramer equation and by using the Laplace transform with the Mittag-Leffler function method, we obtain the analytical solutions of the two fractional equations. Finally, the predicted traveling pulse solutions for each case at different values of fractional order are illustrated in section 3.

2. The Klein-Kramer equation and its approximations

From the Langevin equation (2), one can derive the corresponding fluctuation-averaged phase space dynamics governed by the Klein-Kramer equation, which describes the Brownian motion of particles with the presence of an external force $F(z)$

$$\frac{\partial W(z,v,t)}{\partial t} = \left[-v\frac{\partial}{\partial z} + \frac{\partial}{\partial v}\left(\eta v - \frac{F(z)}{m}\right) + \frac{\eta k_B T}{m}\frac{\partial^2}{\partial v^2} \right] W(z,v,t), \tag{3}$$

where $W(z,v,t)$ is the distribution function of energetic particles, k_B is Boltzmann constant, and T is the absolute temperature. The generalization to anomalous diffusion behavior leads to the fractional Klein-Kramer equation

$$D_t^\alpha W(z,v,t) = \left[-vD_z^\beta + \frac{\partial}{\partial v}\left(\eta v - \frac{F(z)}{m}\right) + \frac{\eta k_B T}{m}\frac{\partial^2}{\partial v^2} \right] W(z,v,t), \tag{4}$$

where D_t^α and D_z^β are the Caputo fractional derivative operator defined in equation (1) for time and space respectively.

2.1. Analytical solution of the fractional modified Telegraph equation

The first approximation of the fractional Klein-Kramer equation is obtained by using the Davies approach [8] and by integrating equation (4) over both $\int .dv$ and $\int .vdv$. A combination of both resulting equations leads to the fractional modified Telegraph equation with an external force

$$\tau^\alpha D_t^\alpha D_t^\alpha w(z,t) + D_t^\alpha w(z,t) = BD_z^\beta w(z,t) + AD_z^\beta D_z^\beta w(z,t), \tag{5}$$

where $w(z,t) = \int_{-\infty}^{\infty} W(z,v,t)\,dv$, $\frac{1}{\eta} = \tau$, $A = \frac{k_B T}{m\eta}$, $B = \frac{-F(z)}{m\eta}$ and F is a constant force.The values of all parameters depend on the comparison with the modified Telegraph equation in Ref. [9] which describes the diffusion of energetic particles for isotropic density case. We introduce a separation ansatz of the form $w(z,t) = \varphi(z)\,T(t)$ together with a choice of initial conditions that is compatible with [9]:

$$T(0) = 1, \quad \partial_t T(0) = 0. \tag{6}$$

We use the Mittag-Leffler function method [10] to solve the spatial part $\varphi(z)$ and Laplace transform to solve $T(t)$. Finally, the general solution of the space time fractional modified Telegraph equation is given by

$$w(z,t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^n \left(\frac{-B \pm \sqrt{B^2 + 4A\lambda}}{2A}\right)^m z^{m\beta}}{n! \, \Gamma(m\beta + 1)} \frac{t^{\alpha n}}{\tau^{\alpha n}} \tag{7}$$

$$\left[\frac{t^{\alpha}}{\tau^{\alpha}} H_{1,2}^{1,1}\left[\lambda \tau^{-\alpha} t^{2\alpha} \left| \begin{array}{c} (-n,1) \\ (0,1), (-n\alpha - \alpha, 2\alpha) \end{array} \right. \right] + H_{1,2}^{1,1}\left[\lambda \tau^{-\alpha} t^{2\alpha} \left| \begin{array}{c} (-n,1) \\ (0,1), (-n\alpha, 2\alpha) \end{array} \right. \right] \right],$$

where λ is the separation constant, which we take as minus unity.

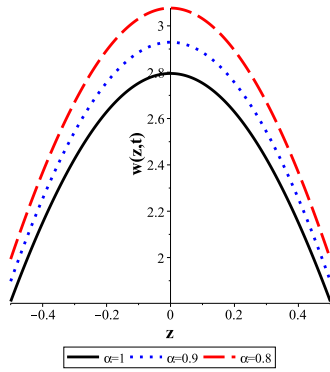**Figure 1.** Predicted travelling pulse solutions of the fractional modified Telegraph equation (7) for different values of α at $\beta = 1$.
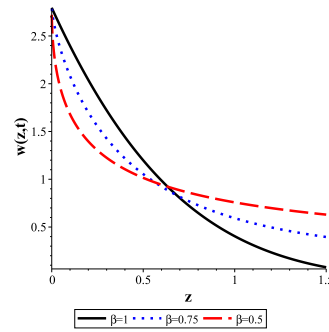
Figure 2. Predicted travelling pulse solutions of the fractional modified Telegraph equation (7) for different values of β at $\alpha = 1$.

2.2. Analytical solution of the fractional Rayleigh equation

To describe the distribution in velocity space averaging out the particle position in the absence of an external force with integration by parts the fractional Klein-Kramer equation (4) will be reduced to the fractional Rayleigh equation

$$D_t^{\alpha} \psi(v,t) = \eta \left[\frac{\partial}{\partial v} v + A\eta \frac{\partial^2}{\partial v^2} \right] \psi(v,t) \tag{8}$$

where $\psi(v,t) = \int_{-\infty}^{\infty} W(z,v,t) \, dz$. The Rayleigh describes the relaxation of the probability distribution function $\psi(v,t)$ toward the stationary Maxwell distribution [11]. To solve the fractional Rayleigh equation (8) consider the function $\psi(v,t)$ to be separated into two parts $\psi(v,t) = F(v)\,\omega(t)$, with the initial value given by

$$\omega(0) = 1, \quad \partial_t \omega(0) = 0. \tag{9}$$

Then, the general solution of the fractional Rayleigh equation reads:

$$\psi(v,t) = \frac{e^{\frac{v^2}{2A\eta}}}{2} E_{\alpha,1}\left(-\lambda t^{\alpha}\right) \left[H\left(\frac{\lambda}{\eta}, \frac{v}{\sqrt{2A\eta}}\right) + F_1\left(\frac{-\lambda}{2\eta}, \frac{1}{2}, \frac{v^2}{2A\eta}\right) \right], \tag{10}$$

where λ is the separation constant, which we take as minus unity, $E_{\alpha,1}$ is the Mittag-Leffler function [12], H is the Hermite function [13] and F_1 is the Hyper-geometric function [14].
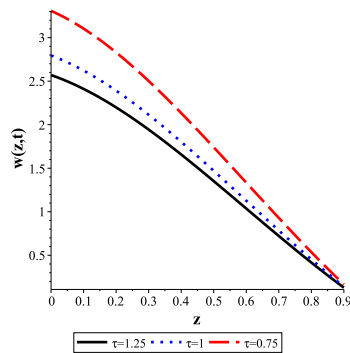
Figure 3. Predicted travelling pulse solutions of the fractional modified Telegraph equation (7) for different values of τ at $\alpha = \beta = 1$.
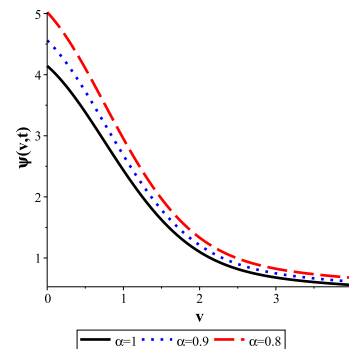

Figure 4. Predicted travelling pulse solutions of the fractional Rayleigh equation (10) for different values of α.

3. Discussion and conclusion

Litvinenko *et al* [9] derived a modified Telegraph equation for isotropic density case, which describes the probability distribution function for the energetic particle transport in the heliosphere. We choose the physical parameters in equation (5) to construct a generalization of the modified telegraph equation to describe the sub-diffusion process of the energetic particles. In figure 1, the solution of the fractional modified telegraph equation illustrates the probability distribution function for both normal diffusion and sub-diffusion processes for different values of time fractional order and attempts to show the accurate prediction on a shorter time scale. Also, the sub-diffusion of the energetic particle is characterized by the non-Gaussian distribution such as in figure 2. Furthermore, in figure 3, we show the change of the probability distribution function for different values of relaxation time. Hence, depending on the parameters, the relaxation time can show either weak or strong z dependence. Finally, in the velocity space, the velocity distribution of the energetic particles in the sub-diffusion limit deviate from the Maxwellian as illustrated in the solution of the fractional Rayleigh equation figure 4.

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