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Energy spectrum inverse problem of q-deformed harmonic oscillator and entanglement of composite bosons

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Abstract. Using the simple deformed three-level model (D3L model) proposed in our early work, we study the entanglement problem of composite bosons. Consider three first energy levels are known, we can get two energy separations, and can define the level deformation parameter δ. Using connection between $q-$deformed harmonic oscillator and Morse-like anharmonic potential, the deform parameter q also can be derived explicitly. Like the Einstein's theory of special relativity, we introduce the observer effects: out side observer (looking from outside the studying system) and inside observer (looking inside the studying system). Corresponding to those observers, the outside entanglement entropy and inside entanglement entropy will be defined.. Like the case of Foucault pendulum in the problem of Earth rotation, our deformation energy level investigation might be useful in prediction the environment effect outside a confined box.

1. Introduction

Deformed Heisenberg algebras with q-deformed harmonic oscillator recently have been attracted a great attention and are a subject of intensive investigation. This approach is found some applications in various branches of physics and chemistry [1-6]. This method of q-deformed quantum mechanics was developed on the base of Heisenberg commutation relations, and the main parameter is the deformation parameter $q \in [0, 1]$.

In the atomic and molecular physics, the Morse potential plays an important role. This potential excellent describes the the interaction between atoms in diatomic and even polyatomic molecules. One-dimensional Morse-like effective potentials with three-parameter set have many application in condensed matter, bio-physics, nano science and quantum optics [7-10].

The relation between the problem of q-deformed harmonic oscillator and Morse potential was considered in [8]. The vibration energy levels of a Morse potential are rather good described by the energy levels of q-deformed harmonic oscillator.

The extended SU(2) model for Morse potential is also developed to compare with phenomenological Dunham's expansion and found a good agreement with experimental data for numerous diatomic molecules [8-10].

Considering deformed algebra is a mathematical object and atomic effective potential likes a physical model, the properties of q-deformed harmonic oscillator on the base of the Morse potential was investigated inversely way in [11, 12]. The infinite equal-step levels of non-deformed harmonic oscillator can be mapped to the energy spectrum of the motion in a parabolic potential. The finite unequal-step levels of q-deformed harmonic oscillator might be described as the motion in a Morse-like anharmonic potential.

In the our previous work [13], the simple deformed three-level model is proposed (D3L model). Consider a three-level system with the new parameter δ characterized the level deformation. The corresponding Morse-like potential can be constructed and then the deformed parameter q can be defined for this system by using the connection between q-deformed harmonic oscillator and Morse-like anharmonic potential.

In this work we continue investigate this D3L model in the frame work of composite boson approach developed in [14-15]. The deformation of energy levels can be characterized by the entanglement entropy between the constituents of composite bosons. Like Einstein's theory of special relativity, we introduce the observer effects: out side observer (looking from outside the studying system) and inside observer (looking inside the studying system). Corresponding to those observers, the outside entanglement entropy and inside entanglement entropy will be defined. Standard cases [14-15] are relating to the outside observer with outside entanglement entropy.

Very recently, gravitational waves from a Binary Black Hole Merger was observed [16] with the LIGO. In the past, the Foucault pendulum told us the rotation of the Earth. That are two examples what inside observers can got the information beyond their systems. That equivalent to the problem how the observer in a closed box know what happen outside the box?. We consider that our D3L model with the case of inside observer and inside entanglement entropy could be useful to this problem.

2. Inverse problem of deformed harmonic oscillator

Consider the first three energy levels E_0, E_1, $E_2$are known, we can define two parameters $\Delta_1 = E_1 - E_0$, and $\Delta_2 = E_2 - E_1$. Introduce the ratio parameter δ as the main parameter of our model [13]

$$\delta = \Delta E_2/\Delta E_1. \tag{1}$$

We have the deformation parameter q

$$q = \frac{1}{2-\delta} \longleftrightarrow \delta = 2 - \frac{1}{q}, \tag{2}$$

the relation between q and δ are presented in the Fig. 1a and Fig. 1b

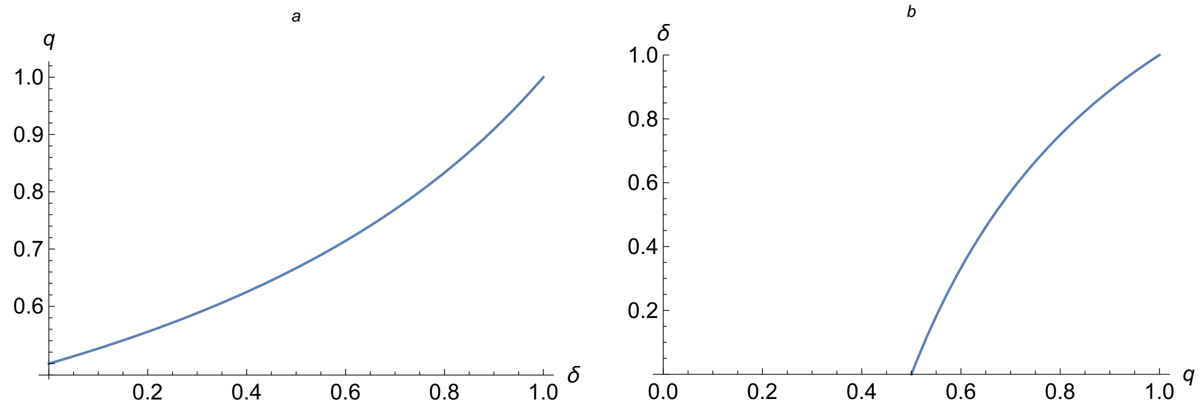

Figure 1. The relation between q and δ: a) q on δ, and b) δ on q.

In the connection between q-deformed harmonic oscillator and Morse-like anharmonic potential, the maximum level number n_{max} is an important parameter [13]. The value n_{max} is

$$n_{max} = \left[\frac{(2-\delta)}{2\,(1-\delta)}\right] = \left[\frac{1}{2\,(1-q)}\right], \tag{3}$$

where [...] is the integer part, and are presented in the Fig. 2 as a functions on δ (upper steps), and on q (lower steps).

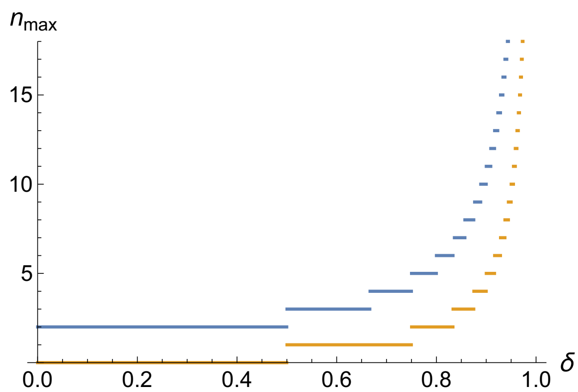

Figure 2. The relation value n_{max} on δ (upper steps), and on q (lower steps).

3. Entanglement entropy of composite bosons

Composite bosons or quasibosons are systems of quasi-particles, which consisted from two or more constituent particles. Among quasibosons in condensed mater and atomic physics there are excitons, atoms, molecules, polaritons, etc. In this part we focus on the case of two component composite bosons.

In order to find the energy dependence of the entanglement entropy we need the expression for the Hamiltonian H_C of the composite boson system with oscillation frequency ω_C. The Hamiltonian of deformed oscillators (deformed bosons) which provide realization of the composite bosons is [14-15]

$$H_C = \sum_n E_{Cn} = \sum_n \frac{1}{2}\hbar\omega_C\left[\varphi\left(n\right) + \varphi\left(n+1\right)\right]. \tag{4}$$

The structure function$\varphi(n)$ involving discrete deformation parameter f is quadratic in n, where n is the number of level energy or number of composite bosons

$$\varphi\left(n\right) = \left(1 + \epsilon\frac{f}{2}\right)n - \epsilon\frac{f}{2}n^2, \tag{5}$$

here $\epsilon = +1$ for fermionic and $\epsilon = -1$ for bosonic constituents, and f is a real number.

Hamiltonian of the composite boson system can be written in the form

$$H_C = \sum_n E_{Cn} = \sum_n \hbar\omega_C \left\{\left(1 + \epsilon\frac{f}{2}\right)\left(n + \frac{1}{2}\right) - \epsilon\frac{f}{2}\left(n + \frac{1}{2}\right)^2 + \epsilon\frac{f}{8}\right\}. \tag{6}$$

Hamiltonian of the inverse problem with the Morse potential representation is

$$H_I = \sum_n E_n = \sum_n \Delta_1 \left[(2 - \delta)\left(n + \frac{1}{2}\right) - (1 - \delta)\left(n + \frac{1}{2}\right)^2\right]. \tag{7}$$

The above two Hamiltonian are same form by replacing

$$\hbar\omega_C \to \Delta_1,\ \epsilon\frac{f}{2} \to 1 - \delta, \tag{8}$$

and neglecting some constant.

The entanglement entropy S_{ent} is defined as

$$S_{ent} = \ln\left(\frac{2}{f}\right). \tag{9}$$

Energy of one composite boson $n = 1$ is

$$E = E_{C1} = \hbar\omega_C\left(\frac{3}{2} - \epsilon\frac{f}{2}\right) \tag{10}$$

Therefore the entanglement entropy $S_{ent}\left(E\right)$ equals

$$S_{ent}\left(E\right) = \ln\left[\frac{\epsilon}{\left(\frac{3}{2} - \frac{E}{\hbar\omega_C}\right)}\right], \tag{11}$$

or for the case of fermion constituents with energy in the interval $\hbar\omega_C/2 \leq E \leq 3\hbar\omega_C/2$

$$S_{ent}\left(E\right) = -\ln\left(\frac{3}{2} - \frac{E}{\hbar\omega_C}\right), \tag{12}$$

and for the case of boson constituents with energy in the interval $3\hbar\omega_C/2 \leq E \leq 5\hbar\omega_C/2$

$$S_{ent}\left(E\right) = -\ln\left(-\frac{3}{2} + \frac{E}{\hbar\omega_C}\right). \tag{13}$$

The entanglement entropy S_{ent} versus the value $\xi = E/\left(\hbar\omega_C\right)$ is presented in the Fig. 3 for two cases $\epsilon = +1$ with $1/2 \leq \xi \leq 3\hbar\omega_C/2$ and $\epsilon = -1$ with $3/2 \leq \xi \leq 5\hbar\omega_C/2$. In both cases, the entropy $S_{ent}\left(E\right)$ goes to infinity for the energy $E = 3\hbar\omega_C/2$, which implies maximal entanglement between constituents.

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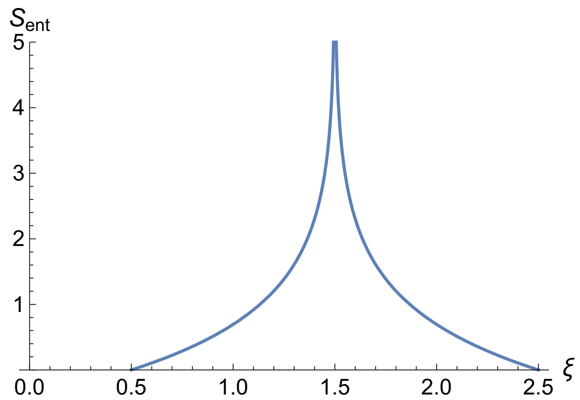

Figure 3. The entanglement entropy S_{ent} versus the value $\xi = E/(\hbar\omega_C)$ for two cases $\epsilon = +1$ with $1/2 \leq \xi \leq 3\hbar\omega_C/2$ and $\epsilon = -1$ with $3/2 \leq \xi \leq 5\hbar\omega_C/2$.

In this case the constituents (fermionic or bosonic) become most tightly bound within a quasiboson, and the quasiboson is most close to pure boson. On the contrary for $E = 3\hbar\omega_C/2$, $\epsilon = +1$, and $E = 5\hbar\omega_C/2$, $\epsilon = -1$, the entanglement entropy $S_{ent} = 0$ i.e. the constituents are unentangled. From physical viewpoint, in this case the constituents are in fact unbound.

4. Inverse problem of deformed harmonic oscillator and entanglement entropy

In order to applicable to the deformed harmonic oscillators we reformulate above problem of entanglement entropy of composite quasi-bosons following way:

(i) Instead of the fermionic or bosonic parameter of the constituents ϵ, we introduce the sign $\eta = \pm 1$ corresponds to the increasing or reducing cases of the energy gaps. Example: energy spectrum of Morse potential with reducing gaps is corresponding to the case $\eta = (+1)$, phonon spectrum in deformed universe model with increasing gaps is corresponding to the case $\eta = (-1)$.

(ii) Assume the observer effect like the Einstein theory of relativity. There are two cases: looking from outside and looking inside. a) Looking from out side case is similar the above case: one composite boson with two constituents. b) Looking from inside case: instead of the composite quasi-bosons with two fermionic or bosonic constituents we investigate one constituent boson entangling with another object.

4.1. Looking from outside case

Using the connection $\eta f/2 \leftrightarrow 1-\delta$ between the two problems, we can define outside entanglement entropy S_O in energy spectrum inverse problem as

$$S_O(\delta) = \ln\left[\frac{\eta}{(1-\delta)}\right], \tag{14}$$

where $0 \leq \delta \leq 0$ for the case of reducing gaps $\epsilon = +$, and $1 \leq \delta$ for the case of increasing gaps $\epsilon = +$.

Using the Morse potential representation of q-deformed harmonic oscillator, we have the connection $1 - \delta \leftrightarrow (1/q) - 1$, the outside entanglement entropy S_I in this case can be expressed in the form

$$S_O(q) = \ln\left[\frac{\eta q}{(1-q)}\right]. \tag{15}$$

The values of outside entanglement entropy S_{O+} versus δ and q are plotted in the Fig. 4a and 4b, respectively.

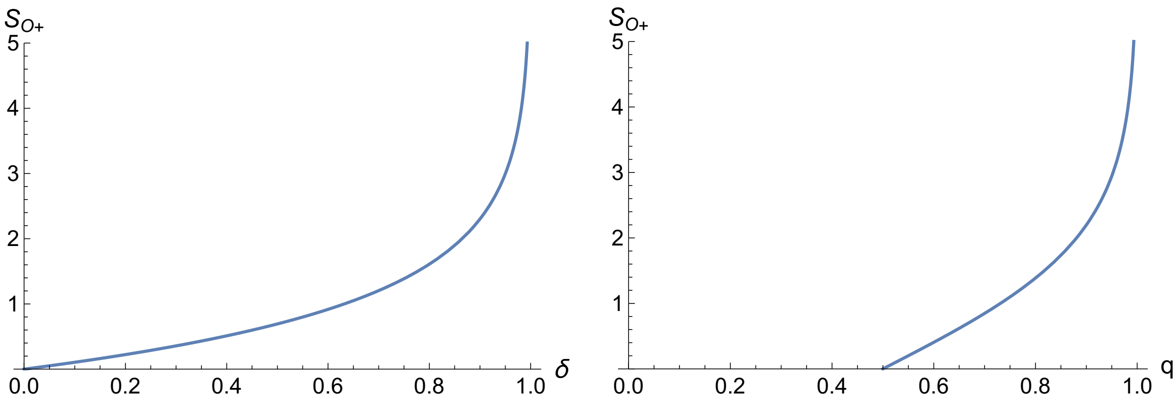

Figure 4. The values of outside entanglement entropy S_{O+} a) versus δ and b) versus q

We can see that outside entanglement entropy goes to infinity $S_{O+} \to \infty$ for $\delta \to 1$ or $q \to 1$ which implies maximal entanglement between two constituents. In this case the constituents (fermionic or bosonic) become tightly bound within a quasiboson, and the quasiboson is most close to pure boson. On the contrary, for $\delta \to 0$ or $q \to 1/2$ (only one energy level is existed), the outside entanglement entropy tents to zero $S_{O+} \to 0$) i.e. the constituents are unentangled. From physical viewpoint, in this case the constituents are in fact unbound.

4.2. Looking from inside case.
We consider a single non-interacting boson is non-deformed boson with equal energy levels $\delta = 1$ (presented by parabolic potential) and its inverse entanglement entropy $S_I = 0$. When a another object (second boson) is appeared near its location to form a composite boson, due to the interaction between the two constituent bosons, the energy spectrum of first boson is changed (presented by Morse potential). We can say the investigate boson recognizes the existence of the other object, this filling can be expressed by the inverse entanglement entropy (or inverse Shannon information)$S_I \neq 0$.

Using again the connection $\eta f/2 \leftrightarrow 1-\delta$ between the two problems, and in analogy the Einstein theory of relativity we can define inside entanglement entropy S_I in energy spectrum inverse problem as

$$S_I\left(\delta\right) = \left(\ln\left[\frac{\eta}{(1-\delta)}\right]\right)^{-1}. \tag{16}$$

Using the Morse potential representation of q-deformed harmonic oscillator, we have the connection $1-\delta \leftrightarrow (1/q)-1$, the inside entanglement entropy S_I in this case can be expressed in the form

$$S_I\left(q\right) = \left(\ln\left[\frac{\eta q}{(1-q)}\right]\right)^{-1}. \tag{17}$$

The values of inside entanglement entropy S_{I+} versus δ and q are plotted in the figure 5, a) and b), respectively.

We can see that inside entanglement entropy goes to zero $S_I \to 0$ for $\delta \to 1$ or $q \to 1$ which implies minimal entanglement between the investigate boson and the object. In this case the

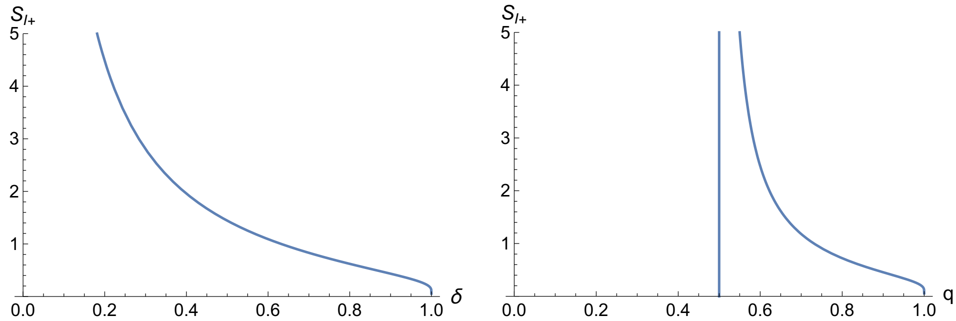

Figure 5. The values of inside entanglement entropy S_{I+} a) versus δ and b) versus q

investigated boson and the object become unbound within a quasiboson or they are not affected one to another. The investigated constituent quasiboson is most close to pure boson. On the contrary, for $\delta \to 0$ or $q \to 1/2$ (only one energy level is existed), the inside entanglement entropy tents to infinity $S_I \to \infty$) i.e. the investigate bonson and out box object are strongly entangled. From physical viewpoint, in this case the investigate boson and object are in tightly bound.

We can use the results of looking inside case to design a seismometer inside a closed box to detecting object outside the box by deformed energy levels.

5. Discussion

In this work using the our simple deformed three-level model (D3L model) proposed in [13], we study the entanglement problem of composite bosons.

Consider three first energy levels E_0, E_1, E_2 are known, we can get two energy separations $\Delta_1 = E_1 - E_0$, and $\Delta_2 = E_2 - E_1$ (consider $\Delta_2 \neq \Delta_1$ in general), and can define the level deformation parameter $\delta = (\Delta E_2/\Delta E_1)$. Using connection between q-deformed harmonic oscillator and Morse-like aharmonic potential, the deform parameter $q = 1/(2-\delta)$ also can be derived explicitly. In the week-deform limit $\delta \to 1$,$q \to 1$, we back to the harmonic case with unique step levels $\Delta_2 = \Delta_1$. In the strong-deform limit $\delta \to 0$, $q \to 1/2$, we go to the two-level problem, where only the ground and first levels can be existed, and the other levels are collapsed, so $\Delta_2 = 0$.

The deformation of energy levels can be characterized by the entanglement entropy between the constituents of composite bosons. Like Einstein's theory of special relativity, we introduce the observer effects: out side observer O (looking from outside the studying system) and inside observer I (looking inside the studying system). Corresponding to those observers, the outside entanglement entropy S_O and inside entanglement entropy S_I will be defined.

Standard cases are relating to the outside observer with outside entanglement entropy. Outside entanglement entropy goes to infinity $S_{O+} \to \infty$ for $\delta \to 1$ or $q \to 1$ which implies maximal entanglement between two constituents. In this case the constituents (fermionic or bosonic) become tightly bound within a quasiboson, and the quasiboson is most close to pure boson. On the contrary, for $\delta \to 0$ or $q \to 1/2$ (only one energy level is existed), the outside entanglement entropy tents to zero $S_{O+} \to 0$) i.e. the constituents are unentangled. From physical viewpoint, in this case the constituents are in fact unbound.

The looking from inside case is interestingly. Inside entanglement entropy goes to zero $S_I \to 0$ for $\delta \to 1$ or $q \to 1$ which implies minimal entanglement between the investigate boson and the object. In this case the investigated boson and the object become unbound within a quasiboson or they are not affected one to another. The investigated constituent quasiboson is most close to pure boson. On the contrary, for $\delta \to 0$ or $q \to 1/2$ (only one energy level is existed), the inside entanglement entropy tents to infinity $S_I \to \infty$) i.e. the investigate boson and out box object are strongly entangled. From physical viewpoint, in this case the investigate boson and object are in tightly bound.

We can use the results of looking inside case to design a seismometer inside a closed box to detecting object outside the box by deformed energy levels. Like the case of Foucault pendulums in the problem of Earth rotation, and gravitation sensors LIGO, our deformation energy level investigation might be useful in prediction the environment effect outside a confined box effect (inside observer and outside effect) by using the inside entanglement entropy approach.

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