

# Rotational stellar structures based on the Lagrangian variational principle

Nobutoshi Yasutake<sup>1</sup>, Kotaro Fujisawa<sup>2</sup>, Shoichi Yamada<sup>2</sup>

<sup>1</sup> Physics Department, Chiba Institute of Technology, Shibazono 2-1-1, Narashino, Chiba, 275-0023, Japan

<sup>2</sup> Advanced Research Institute for Science and Engineering, Waseda University, Okubo 3-4-1, Shinjuku, Tokyo 169-8555, Japan

E-mail: <sup>1</sup>nobutoshi.yasutake@p.chibakoudai.jp

**Abstract.** A new method for multi-dimensional stellar structures is proposed in this study. As for stellar evolution calculations, the Heney method is the defacto standard now, but basically assumed to be spherical symmetric. It is one of the difficulties for deformed stellar-evolution calculations to trace the potentially complex movements of each fluid element. On the other hand, our new method is very suitable to follow such movements, since it is based on the Lagrange coordinate. This scheme is also based on the variational principle, which is adopted to the studies for the pasta structures inside of neutron stars. Our scheme could be a major break through for evolution calculations of any types of deformed stars: proto-planets, proto-stars, and proto-neutron stars, etc.

## 1. Introduction

Stellar evolution is a mature topic in astrophysics. After the seminal works [1-4], the Heney method [5] is the defacto standard now for stellar evolution calculations. Although this method is proposed originally for spherical stars without rotation and magnetic field, there are some subspecific techniques taking into account the effects of rotation and/or magnetic field [6]. Such subspecific techniques are, however, based on the spherical symmetric assumption, and no one succeeds to propose a new scheme for fully-multi-dimensional stellar-evolution calculations.

One of the difficulties of fully-multi-dimensional stellar-evolution calculations is to obtain hydrostatic equilibria of deformed stars considering “*baroclinicity*”, in which pressure depends not only on density but also temperature and chemical composition. It is necessary when we adopt a realistic equation of state (EoS) in stellar evolution calculations. Note that we can neglect the baroclinicity for neutron stars, because thermal effects on pressure are negligible, and chemical composition is derived uniquely from the beta equilibrium condition and neutrino free condition. Such cases are known as “*barotropic*” ones, in which pressure depends only on density. On the other hand, the baroclinicity appears in proto-neutron stars.

Even if some equilibria are obtained, it is highly non-trivial whether we can make a sequence of them that represents stellar evolutions appropriately, because the fluid elements composing rotating stars change their positions non-radially in complicated ways. In this study, we have introduced the Lagrange scheme to trace the moving of fluid elements in conservative ways. Otherwise, in a Eulerian scheme, it would be difficult to describe such displacements. This



difficulty appears in both barotropic cases (neutron stars) and baroclinic cases (proto-neutron stars).

It is not a new attempt to adopt the Lagrange scheme into multi-dimensional calculation, indeed already Freedman & Shutz have pointed out that trivial Lagrange displacements do not provide unique solution because of the gauge degree of freedom [7]. In this study, we solve this problem by using the Monte Carlo method.

## 2. Formalism

Let us introduce an idea of the Lagrangian variational principle. The details are shown in our papers [8, 9].

First we prepare nodes, which are connected with each other by edges making triangulations. Each node has mass, angular momentum, entropy, and compositions, which are conserved during the process to obtain hydro-static equilibria. We do not take into account energy equation, and just focus on the balance equation assuming pure hydrogen matter for simplicity, since this study is the preliminary step. Note that there is no problem to take account realistic composition in our scheme. It however changes just the pressure (EoS) slightly in our current formalism. If we considered nuclear burnings, advection of matter, and convections, etc., the composition would have a critical role. It remains as a future task on us.

Second, the finite volume is calculated by the triangulations around each node, then density, pressure, temperature, internal energy, etc. are derived. Finally, we obtain the most optimal structures changing the positions of all nodes to find the minimal energy. We adopt the Monte Carlo method in the iterative process: we change the positions of nodes gradually, one by one, at random. Note that periodical smoothing is necessary to avoid a stagnancy, in which the system is stucked with local minimum energy.

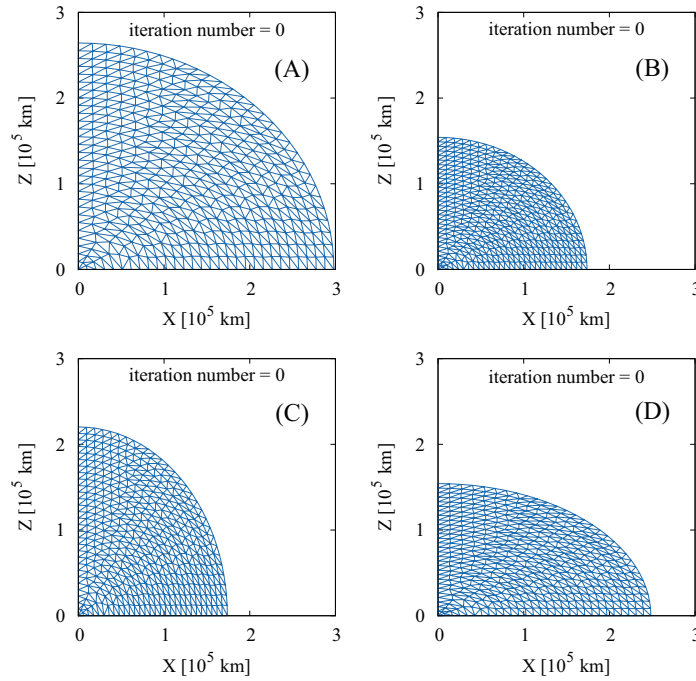
The idea of this work is quite simple and similar to the studies of *the pasta-structures*, which are the deformed nuclear-structures in neutron stars. In Table 1, we show the duality between hydrostatic equilibria and the pasta structures.

**Table 1.** Duality between deformed stellar structures and “*pasta*” structures in the crust of NSs, or in the quark-hadron phase transition in the core of NSs.

	Deformed stellar structures	Pasta structures
Basic equations	<ul style="list-style-type: none"> <li>· Hydro-static equilibrium</li> <li>· Poisson equation (gravitation)</li> <li>· Equation of state</li> </ul>	<ul style="list-style-type: none"> <li>· Chemical equilibrium</li> <li>· Poisson equation (Coulomb interaction)</li> <li>· Equation of state</li> </ul>
Method	Variational principle in Lagrange coordinate	Variational principle in Euler coordinate
Origin of deformation	Balance between the repulsive force of rotation, pressure and the attractive force by gravitation	Balance between the repulsive force of Coulomb interaction, pressure and the attractive force by surface tension
Reference	[8][9]	[10]

## 3. Results

Let us show some examples of obtained hydrostatic equilibria with baroclinic cases in this section.



**Figure 1.** Initial models. Details are described in the text.

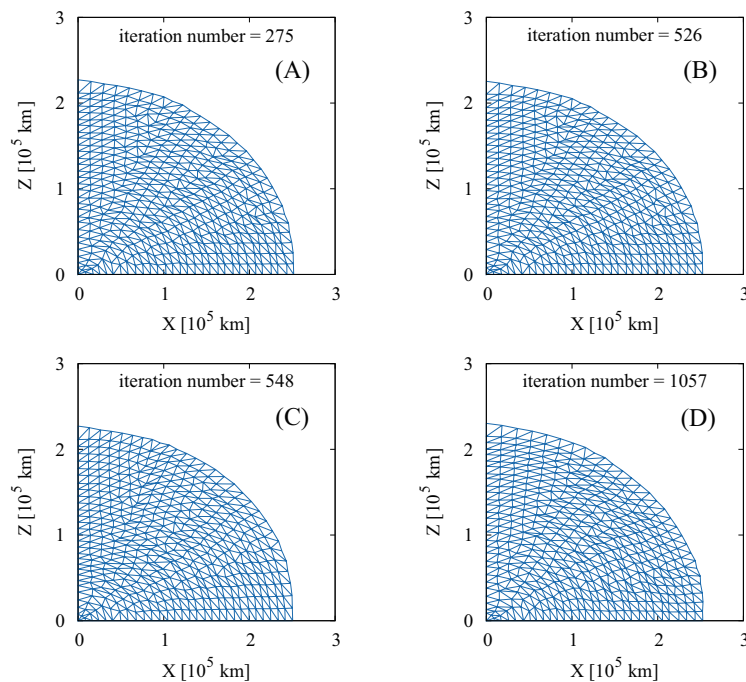
As described in the last section, first we need to prepare some profiles of mass, angular momentum, and entropy. Here, we introduce four types of initial models to check whether the resultant solutions are free or not from the local energy minima, since our method is based on the variational principle. These initial models are shown in Fig. 1. We extend 30 % radially an solution, which is given by Fujisawa method [11], and name it as initial model (A). For other three models, we shrink the solution by 30 % in three different ways: (B) radially, (C) horizontally, and (D) vertically. The rotation profile of the solution is given on the equatorial axis as following,

$$\Omega^2 = \frac{\Omega_0^2}{(\varpi^2/R_e^2 + d^2)}, \quad (1)$$

where the arbitral parameter  $\Omega_0$ ,  $R_e$  and  $d$  are set as  $\Omega_0 = 1.9 \times 10^{-3} \text{ rad s}^{-1}$ ,  $R_e = 2.57 \times 10^5 \text{ km}$ , and  $d = 0.90$  respectively. This value of  $\Omega$  is a few hundred times the solar angular velocity. Here,  $\varpi$  is distance from the rotational axis. The rotation profile of the solution in other area is obtained self-consistently to satisfy the entropy distribution given as following,

$$K = K_0 \left\{ 1 + e_1 (1 + e_2 P_2(\cos \theta)) \frac{r^2}{R_e^2} \right\}, \quad (2)$$

where  $r$  and  $K$  is the distance from center and position-dependent value in the polytrope-like EoS,  $p = K\rho^\gamma$  ( $p$  and  $\rho$  are pressure and density, and  $\gamma$  is fixed as  $\gamma = 5/3$  in this study). The Legendre polynomials are denoted by  $P_n$  with  $\theta$  being colatitude. The dimensionless parameter  $e_1$  specifies radial non-uniformity whereas  $e_2$  gives colatitudinal dependency, set as  $e_1 = 0.45$  and  $e_2 = 0.80$  here. The values of  $K_0$  is chosen to be  $5.49 \times 10^{13}$  in cgs unit, respectively. These models are highly artificial admittedly, but it is known that such entropy distribution gives the rotation profile of the shellular type, which has an onion-skin-like angular velocity profile, known as the standard type based on the Zahn's conjecture [12].



**Figure 2.** Final results corresponding to the initial models shown in Fig. 1.

The resultant structures for models (A)-(D) are shown in Fig. 2. Clearly, they look similar, though the initial models are extremely different from each other. This means that these solutions are free from the local energy minima. We have also checked that the results are consistent with the Bjerknes-Rosseland rules, and do not break the Solberg-Hoiland criterion.

#### 4. Discussion & Summary

We study hydro-static equilibria for baroclinic cases using the Lagrangian variational principle and the Monte Carlo method. The solutions are free from the local energy minima. Our method could be applicable to stellar evolution calculations, since we can trace the displacements of fluid elements in conservative ways, although we do not take into account convection for these models.

Possible applications of our scheme will be extended to compact stars, proto-stars and proto-planets to mention a few. Magnetic fields and/or general relativity should be considered in due course.

#### Acknowledgments

We are grateful to K. Suzuki, T. Yamasaki, M. Okamoto, and T. Maruyama for fruitful discussions. This work was partially supported by JSPS KAKENHI Grant Numbers 25105510, 24244036 and Grant-in-Aid for Scientific Research on Innovative Areas, No.24103006.

#### References

- [1] Burbidge K M, Burbidge G R, Fowler W A, Hoyle F 1957 *Review of Modern Physics* **29** 4
- [2] Cameron A G W 1957 *Chalk River Report* CRL-41
- [3] Hayashi C 1961 *PASJ* **13** 442
- [4] Hayashi C 1961 *PASJ* **13** 450
- [5] Henyey L G, Forbes J E and Gould N L 1964 *ApJ* **139** 306
- [6] Woosley S E, Heger A, and Weaver T A 2002, *Reviews of Modern Physics* **74** 1015 and references therein
- [7] Friedman J L, and Schutz B F 1978 *ApJ* **221** 937

- [8] Yasutake N, Fujisawa K, Yamada S 2015 *MNRAS letter* **446** L56
- [9] Yasutake N, Fujisawa K, Yamada S 2016 *MNRAS* **463** 3705
- [10] Yasutake N, Łastowiecki, BeniĆ S, Blaschke D, Maruyama T, Tatsumi T 2014 *Phys. Rev. C* **89** 065803 and references therein
- [11] Fujisawa K 2015 *MNRAS* **454** 3060
- [12] Zahn J-P 1992 *Astro. Astrophys.* **265** 115