

# On The Partition Dimension of $C_m + P_n$ Graph

Hidra Vertana, Tri Atmojo Kusmayadi

Department of Mathematics, Faculty of Mathematics and Natural Sciences,  
Universitas Sebelas Maret, Surakarta, Indonesia

E-mail: hidravertana@gmail.com, tri.atmojo.kusmayadi@gmail.com

**Abstract.** Let  $G$  be a connected graph with  $V(G) = \{v_1, v_2, \dots, v_i\}$  and  $E(G) = \{e_1, e_2, \dots, e_j\}$ , where  $V(G)$  is vertex set and  $E(G)$  is edge set. If  $S \subseteq V(G)$  and  $v \in V(G)$ , then the distance between  $v$  and  $S$  is defined by  $d(v, S) = \min\{d(v, x) | x \in S\}$ . For an ordered  $k$ -partition  $\Pi = \{S_1, S_2, \dots, S_k\}$  of  $V(G)$ , the representation of  $v$  with respect to  $\Pi$  is  $r(v|\Pi)$  with  $r(v|\Pi) = (d(v, S_1), d(v, S_2), \dots, d(v, S_k))$ . If the representation of  $v \in V(G)$  with respect to  $\Pi$  are distinct, so  $\Pi$  is called a resolving partition of  $V(G)$ . The minimum cardinality of resolving partition  $\Pi$  is called a partition dimension of  $G$ , denoted by  $pd(G)$ . In this paper, we study the partition dimension of a  $C_m + P_n$  graph.  $C_m + P_n$  graph is a graph formed from join operation of cycle graph  $C_m$  with order  $m \geq 3$  and path  $P_n$  with order  $n \geq 2$ .  $C_m + P_n$  is the union  $C_m \cup P_n$  together with all edges  $u_a v_b$ , for  $u_a \in V(C_m)$  and  $v_b \in V(P_n)$  with  $1 \leq a \leq m$  and  $1 \leq b \leq n$ . We obtain the partition dimension of  $C_m + P_n$  graph is  $pd(C_m + P_n) = g$  where  $g$  is the smallest positive integer such that  $n \leq 5g - 12$  for  $g = 5$  and  $n \leq \frac{g^3 - 7g^2 + 20g - 18}{2}$  for  $g \geq 6$ , and  $pd(C_q + P_n) = \min\{p + f, r + t, x + y\}$  for  $q \geq 4$  and  $n \geq 2$  where  $p, f, r, t, x$  and  $y$  are some positive integers related to the number of partition classes containing vertices of  $C_q$  and  $P_n$ .

## 1. Introduction

Graph theory is a branch of mathematics that developed rapidly. In daily life, the graph theory is used to solve some problems with vertex as discrete objects and edge as the relationship between both of them. One of the interesting concepts in graph theory is so called the partition dimension. The partition dimension of a graph was introduced by Chartrand et al. [1] in 1998. The partition dimension is a variation of the metric dimension introduced by Slater in 1975 (Chartrand et al. [3]).

Let  $G$  be a connected graph. For a subset  $S$  of  $V(G)$  and a vertex  $v$  of  $G$ , the distance between  $v$  and  $S$  is defined as  $d(v, S) = \min\{d(v, x) | x \in S\}$ . For an ordered  $k$ -partition  $\Pi = \{S_1, S_2, \dots, S_k\}$  of  $V(G)$  and a vertex  $v$  of  $G$ , the representation  $v$  with respect to  $\Pi$  is  $k$ -vectors  $r(v|\Pi) = (d(v, S_1), d(v, S_2), \dots, d(v, S_k))$ . The partition  $\Pi$  is called a resolving partition if the  $k$ -vectors  $r(v|\Pi)$  are distinct, for every  $v \in V(G)$ . The minimum cardinality of the  $k$ -partition of  $V(G)$  is called the partition dimension of  $G$  denoted by  $pd(G)$ . (Chartrand et al. [2])

Many researches on the partition dimension have applied for specific graph classes. In 2000, Chartrand et al. [2] determined the partition dimension of some graph classes which result  $pd(G) = 2$  if and only if  $G = P_n$ ,  $pd(G) = n$  if and only if  $G = K_n$ , and  $pd(G) = 3$  if  $G = C_n$  for  $n \geq 3$ . In 2007, Tomescu et al. [5] formed the partition dimension of a wheel graph. Hidayat [4] in 2015 determined the partition dimension of some graph classes, one of them is a



$K_1 + (P_m \odot K_n)$  graph. In this paper, we determine the partition dimension of a  $C_m + P_n$  graph for  $m \geq 3$  and  $n \geq 2$ .

## 2. Partition Dimension of $C_m + P_n$ Graph

Let  $C_m + P_n$  be a graph formed from join operation of cycle graph with order  $m$  ( $C_m$ ) and path with order  $n$  ( $P_n$ ).  $C_m + P_n$  is a graph with  $V(C_m + P_n) = V(C_m) \cup V(P_n)$  and  $E(C_m + P_n) = E(C_m) \cup E(P_n) \cup \{u_a v_b | u_a \in V(C_m) \text{ and } v_b \in V(P_n)\}$  with  $1 \leq a \leq m$  and  $1 \leq b \leq n$ . The vertex set of  $C_m + P_n$  graph is  $V(C_m + P_n) = \{u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n\}$  and the number of vertices in a  $C_m + P_n$  graph is  $|V(C_m + P_n)| = m + n$  for  $m \geq 3$  and  $n \geq 2$ . For example of  $C_m + P_n$  graph is in Figure 1.

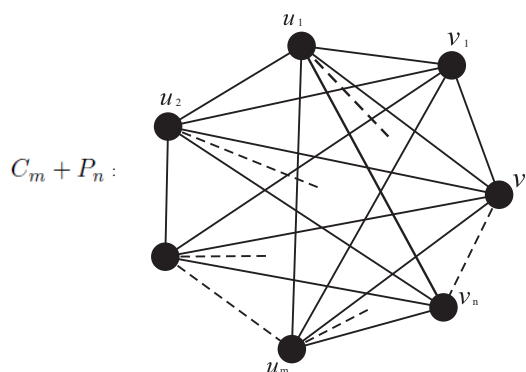


Figure 1.  $C_m + P_n$  Graph

**Lemma 2.1** Let  $C_3 + P_n$  be a graph with  $n \geq 2$ , and let  $\Pi$  be a resolving partition of  $C_3 + P_n$ . If vertex  $u_1, u_2, u_3 \in V(C_3)$ ,  $v_b \in V(P_n)$  for  $b \in [1, n]$ ,  $u_1, v_c \in S_1$ ,  $u_2, v_d \in S_2$ ,  $u_3, v_e \in S_3$  ( $1 \leq c \neq d \neq e \leq n$ ), and  $v_y \in S_j$  for vertex  $v_y \in V(P_n) \setminus \{v_c, v_d, v_e\}$  and  $j = 4, 5, \dots, g$  then  $|S_i| \leq 1 + \binom{g-3}{0} + \binom{g-3}{1}$  for  $i = 1, 2, 3$ ,  $g = 5$  and  $|S_i| \leq 1 + \binom{g-3}{0} + \binom{g-3}{1} + \binom{g-3}{2}$  for  $i = 1, 2, 3$ ,  $g \geq 6$ .

**Proof.** Let  $\Pi = \{S_1, S_2, \dots, S_g\}$  be a resolving partition of  $C_3 + P_n$ . If vertex  $u_1, u_2, u_3 \in V(C_3)$ ,  $v_b \in V(P_n)$  for  $b \in [1, n]$ ,  $u_1, v_c \in S_1$ ,  $u_2, v_d \in S_2$ ,  $u_3, v_e \in S_3$  ( $1 \leq c \neq d \neq e \leq n$ ), and  $v_y \in S_j$  for vertex  $v_y \in V(P_n) \setminus \{v_c, v_d, v_e\}$  and  $j = 4, 5, \dots, g$ , then we obtain the representation  $r(u_1|\Pi) = (0, 1, 1, \dots, 1)$ ,  $r(u_2|\Pi) = (1, 0, 1, \dots, 1)$ ,  $r(u_3|\Pi) = (1, 1, 0, \dots, 1)$ , and  $r(v_c|\Pi) = (0, 1, 1, \dots)$ . Since the diameter of  $C_3 + P_n$  is 2, then other than the three positions of  $r(v_c|\Pi)$  can be filled by 1 or 2. If  $g = 5$  then there are at most one 1 other than the first three positions, the rest can be filled by 2. If  $g \geq 6$  then there are at most two 1's other than the first three positions, then the rest can be filled by 2's. Because vertex  $u_1$  has a single representation, so there are at most  $1 + \binom{g-3}{0} + \binom{g-3}{1}$  distinct representation for  $g = 5$  or at most  $1 + \binom{g-3}{0} + \binom{g-3}{1} + \binom{g-3}{2}$  distinct representation for  $g \geq 6$ . Thus, we obtain  $|S_i| \leq 1 + \binom{g-3}{0} + \binom{g-3}{1}$  for  $i = 1, 2, 3$ ,  $g = 5$  and  $|S_i| \leq 1 + \binom{g-3}{0} + \binom{g-3}{1} + \binom{g-3}{2}$  for  $i = 1, 2, 3$ ,  $g \geq 6$ .  $\square$

**Lemma 2.2** Let  $C_3 + P_n$  be a graph with  $n \geq 2$ , and let  $\Pi$  be a resolving partition of  $C_3 + P_n$ . If the vertex  $u_1, u_2$  or  $u_3$  contained in partition classes  $S_1, S_2$ , or  $S_3$ , then  $|S_j| \leq \binom{g-4}{0} + \binom{g-4}{1}$  for  $j = 4, 5$ ,  $g = 5$  and  $|S_j| \leq \binom{g-4}{0} + \binom{g-4}{1} + \binom{g-4}{2}$  for  $j = 4, 5, \dots, g$ ,  $g \geq 6$ .

**Proof.** Let  $\Pi = \{S_1, S_2, \dots, S_g\}$  be a resolving partition of  $C_3 + P_n$ . Consider the partition class other than the  $S_1, S_2$ , and  $S_3$ . Suppose the vertex  $u_1, u_2$  or  $u_3$  are in  $S_1, S_2$  or  $S_3$  respectively and

vertex  $v_y \in V(C_3 + P_n) \setminus \{u_1, u_2, u_3\}$  for  $1 \leq y \neq c \neq d \neq e \leq n$  is in  $S_4$ , then the representation  $r(v_y|\Pi) = (1, 1, 1, 0, \dots)$ . Therefore  $g-4$  positions on the representation of vertex  $v_y$  can be filled at most one 1 for  $g = 5$  and at most two 1's for  $g \geq 6$ . There are at most  $\binom{g-4}{0} + \binom{g-4}{1}$  distinct representation for  $g = 5$ , or at most  $\binom{g-4}{0} + \binom{g-4}{1} + \binom{g-4}{2}$  distinct representation for  $g \geq 6$ . So,  $|S_j| \leq \binom{g-4}{0} + \binom{g-4}{1}$  for  $j = 4, 5$ ,  $g = 5$  and  $|S_j| \leq \binom{g-4}{0} + \binom{g-4}{1} + \binom{g-4}{2}$  for  $j = 4, 5, \dots, g$ ,  $g \geq 6$ .  $\square$

**Theorem 2.3** Let  $C_3 + P_n$  be a graph with  $n \geq 2$ , the partition dimension of  $C_3 + P_n$  is  $pd(C_3 + P_n) = g$  with  $g$  is the smallest positive integer such that  $n \leq 5g - 12$  for  $g = 5$  and  $n \leq \frac{g^3 - 7g^2 + 20g - 18}{2}$  for  $g \geq 6$ .

**Proof.** Let  $\Pi = \{S_1, S_2, \dots, S_g\}$  be resolving partition of  $C_3 + P_n$  graph for  $n \geq 2$ . By Lemma 2.1 we have  $|S_i| \leq 1 + \binom{g-3}{0} + \binom{g-3}{1}$  for  $i = 1, 2, 3$ ,  $g = 5$  and  $|S_i| \leq 1 + \binom{g-3}{0} + \binom{g-3}{1} + \binom{g-3}{2}$  for  $i = 1, 2, 3$ ,  $g \geq 6$ . By Lemma 2.2 we have  $|S_j| \leq \binom{g-4}{0} + \binom{g-4}{1}$  for  $j = 4, 5$  and  $|S_j| \leq \binom{g-4}{0} + \binom{g-4}{1} + \binom{g-4}{2}$  for  $j = 4, 5, \dots, g$ ,  $g \geq 6$ . Since the number of vertex in  $C_3 + P_n$  graph is  $|V(C_3 + P_n)| = n + 3$ , so

$$\begin{aligned} |V(C_3 + P_n)| &= |S_i| + |S_j|, \\ &\leq (1 + \binom{g-3}{0} + \binom{g-3}{1})(3) + (\binom{g-4}{0} + \binom{g-4}{1})(g-3) \\ n &\leq 5g - 12, \text{ for } g=5. \end{aligned}$$

For  $g \geq 6$  by the same way, we obtain  $n \leq \frac{g^3 - 7g^2 + 20g - 18}{2}$ . Thus,  $pd(C_3 + P_n) = g$  where  $g$  is the smallest positive integer such that  $n \leq 5g - 12$  for  $g = 5$  and  $n \leq \frac{g^3 - 7g^2 + 20g - 18}{2}$  for  $g \geq 6$ .  $\square$

Let  $u_a \in V(C_q)$  for  $a \in [1, q]$  and  $v_b \in V(P_n)$  for  $b \in [1, n]$  be vertices of  $C_q + P_n$  with  $q \geq 4$ ,  $n \geq 2$  and let  $\Pi = \{S_1, S_2, \dots, S_g\}$  be a resolving partition of  $C_q + P_n$ . We have the following remark.

**Remark 2.4**

- (i)  $p$  is a number of partition class containing a vertex  $v_b \in V(P_n)$  such that  $v_b \in S_i$  for  $1 \leq i \leq p$ , and  $u_h \in S_1$ ,  $u_w \in S_2$  for  $u_h, u_w \in V(C_q)$ ,  $1 \leq h \neq w \leq q$ ,
- (ii)  $f = g - p$  is a number of partition class containing a vertex  $u_z \in V(C_q) \setminus \{u_h, u_w\}$ ,  $1 \leq z \neq h \neq w \leq q$ ,
- (iii)  $r$  is a number of partition class containing a vertex  $v_b \in V(P_n)$  such that  $v_b \in S_i$  for  $1 \leq i \leq r$ , and  $u_h \in S_1$  for  $1 \leq h \leq q$ ,
- (iv)  $t = g - r$  is a number of partition class containing a vertex  $u_z \in V(C_q) \setminus \{u_h\}$  for  $1 \leq z \neq h \leq q$ ,
- (v)  $x$  is a number of partition class containing a vertex  $v_b \in V(P_n)$ , and
- (vi)  $y = g - x$  is a number of partition class containing a vertex  $u_a \in V(C_q)$

**Lemma 2.5** Let  $C_q + P_n$  be a graph with  $q \geq 4$  and  $n \geq 2$ , let  $\Pi = \{S_1, S_2, \dots, S_p, S_{p+1}, \dots, S_g\}$  be a resolving partition of  $C_q + P_n$ . If  $v_b \in S_i$  for  $1 \leq i \leq p$ ,  $u_h \in S_1$  and  $u_w \in S_2$  for  $1 \leq h \neq w \leq q$ , then  $n \leq 2$  for  $p = 2$ ,  $n \leq 2p - 1$  for  $p = 3$ ,  $n \leq p^2 - p$  for  $p = 4$  and  $n \leq \frac{p^3 - 5p^2 + 12p - 8}{2}$  for  $p \geq 5$ .

**Proof.** Let  $\Pi = \{S_1, S_2, \dots, S_p, \dots, S_g\}$  be a resolving partition of  $C_q + P_n$  with  $q \geq 4$  and  $n \geq 2$ . Let  $v_b \in S_i$  with  $i = 1, 2, \dots, p$  and a vertex  $u_h \in S_1$  and  $u_w \in S_2$  with  $1 \leq h \neq w \leq q$ . We obtain  $r(u_h|\Pi) = (0, 1, \dots)$ ,  $r(u_w|\Pi) = (1, 0, \dots)$ . Let  $v_c \in S_1$  and  $v_d \in S_2$  with  $1 \leq c \neq d \leq n$ , we obtain  $r(v_c|\Pi) = (0, 1, \dots, 1, \dots, 1)$  and  $r(v_d|\Pi) = (1, 0, \dots, 1, \dots, 1)$ . Since the diameter of  $C_q + P_n$  graph is 2, the elements on  $r(v_c|\Pi)$  other than the first and second positions can be

filled by 1 or 2. There are at most two 1's on representations other than the first and second positions. Therefore,  $p - 2$  positions on the representation  $r(v_c|\Pi)$  can be filled by at most two 1's and the rest can be filled by 2. Let  $p = 2$ , there are at most  $\binom{p-2}{0}$  distinct representations of  $v_c \in S_1$ . Then, for  $p = 3$  there are at most  $\binom{p-2}{0} + \binom{p-2}{1}$  distinct representations of  $v_c \in S_1$ , and let  $p \geq 4$  there are at most  $\binom{p-2}{0} + \binom{p-2}{1} + \binom{p-2}{2}$  distinct representation of  $v_c \in S_1$ . Similarly for  $v_d \in S_2 \setminus \{u_w\}$ . Furthermore, let  $v_y \in S_i \setminus \{S_1, S_2\}$  with  $1 \leq y \neq c \neq d \leq n$ , we obtain  $r(v_y|\Pi) = (1, 1, 0, \dots, 1, \dots, 1)$ . There are at most  $p - 3$  positions on the representation of  $v_y \in S_i \setminus \{S_1, S_2\}$  can be filled with at most two 1's and the rest can be filled by 2. Let  $p = 3$ , there are at most  $\binom{p-3}{0}$  distinct representations of  $v_y \in S_i \setminus \{S_1, S_2\}$ . Then for  $p = 4$ , there are at most  $\binom{p-3}{0} + \binom{p-3}{1}$  distinct representations of  $v_y \in S_i \setminus \{S_1, S_2\}$ , and for  $p \geq 5$  there are at most  $\binom{p-3}{0} + \binom{p-3}{1} + \binom{p-3}{2}$  distinct representation of  $v_y \in S_i \setminus \{S_1, S_2\}$ . Since the number of  $v_b \in V(P_n)$  is  $n$ , then

(i) for  $p = 2$ ,

$$\begin{aligned} |V(P_n)| &= |v_c \in S_1| + |v_d \in S_2| \\ &\leq \binom{p-2}{0} + \binom{p-2}{0} \\ n &\leq 2, \end{aligned}$$

(ii) for  $p = 3$ ,

$$\begin{aligned} |V(P_n)| &= |v_c \in S_1| + |v_d \in S_2| + |v_y \in S_3| \\ &\leq ((\binom{p-2}{0} + \binom{p-2}{1}))(2) + \binom{p-3}{0} \\ n &\leq 2p - 1, \end{aligned}$$

(iii) for  $p = 4$ ,

$$\begin{aligned} |V(P_n)| &= |v_c \in S_1| + |v_d \in S_2| + |v_y \in S_i|, (i = 3, 4) \\ &\leq ((\binom{p-2}{0} + \binom{p-2}{1} + \binom{p-2}{2}))(2) + ((\binom{p-3}{0} + \binom{p-3}{1}))(2) \\ n &\leq p^2 - p, \end{aligned}$$

(iv) for  $p \geq 5$

$$\begin{aligned} |V(P_n)| &= |v_c \in S_1| + |v_d \in S_2| + |v_y \in S_i|, (3 \leq i \leq p) \\ &\leq ((\binom{p-2}{0} + \binom{p-2}{1} + \binom{p-2}{2}))(2) + ((\binom{p-3}{0} + \binom{p-3}{1} + \binom{p-3}{2}))(p-2) \\ n &\leq \frac{p^3 - 5p^2 + 12p - 8}{2}. \end{aligned}$$

So, if  $v_b \in S_i$  for  $1 \leq i \leq p$ ,  $u_h \in S_1$  and  $u_w \in S_2$  for  $1 \leq h \neq w \leq q$ , then  $n \leq 2$  for  $p = 2$ ,  $n \leq 2p - 1$  for  $p = 3$ ,  $n \leq p^2 - p$  for  $p = 4$  and  $n \leq \frac{p^3 - 5p^2 + 12p - 8}{2}$  for  $p \geq 5$ .

**Lemma 2.6** Let  $C_q + P_n$  be a graph with  $q \geq 4$  and  $n \geq 2$ , let  $\Pi = \{S_1, S_2, \dots, S_p, S_{p+1}, \dots, S_g\}$  be a resolving partition of  $C_q + P_n$ . If  $v_b \in S_i$  for  $1 \leq i \leq p$ ,  $u_z \in V(C_q) \setminus \{u_h, u_w\}$  for  $1 \leq z \neq h \neq w \leq q$  contained in the partition class  $S_j$  for  $j = 1, 2, \dots, f$  with  $f = g - p$ , then  $q \leq 2f^2 + f$  for  $n = 2, 5, 12$  and  $f = 2$  and  $q \leq \frac{f^3 + f^2 + 4f + 4}{2}$  for another  $p, f$ .

**Proof.** Let  $\Pi = \{S_1, S_2, \dots, S_p, \dots, S_g\}$  be a resolving partition of  $C_q + P_n$  graph with  $q \geq 4$  and  $n \geq 2$ . Let  $v_b \in V(P_n)$ ,  $v_b \in S_i$  for  $i = 1, 2, \dots, p$ ,  $u_h \in S_1$  and  $u_w \in S_2$ ,  $u_z \in S_j$  with  $u_z \in V(C_q) \setminus \{u_h, u_w\}$ ,  $1 \leq p \neq w \neq z \leq q$  and  $j = 1, 2, \dots, f$  with  $f = g - p$ . From Lemma 2.5, we obtain the representations of  $u_h$  and  $u_w$  with respect to  $\Pi$  are  $r(u_h|\Pi) = (0, 1, 1, \dots, 1, \dots)$

and  $r(u_w|\Pi) = (1, 0, 1, \dots, 1, \dots)$ . Because the diameter of  $C_q + P_n$  graph is 2, the element of  $r(u_h|\Pi)$  other than the first position and  $p - 1$  positions can be filled by 1 and 2 so that the  $g - p$  position on the representation of  $u_h \in S_1$  can be filled at most two 1's and the rest can be filled by 2. There are at most  $\binom{g-p}{0} + \binom{g-p}{1} + \binom{g-p}{2}$  distinct representations of  $u_h \in S_i$ . Since  $f = g - p$ , so it can be written  $\binom{f}{0} + \binom{f}{1} + \binom{f}{2}$  distinct representations of  $u_h \in S_1$ . Suppose  $n = 2, 5, 12$  and  $f = 2$ , there are two equal representation  $r(v_c|\Pi) = r(u_h|\Pi)$  so there at most  $\binom{f}{0} + \binom{f}{1} + \binom{f}{2} - 1$  distinct representation for all  $u_h \in S_1$ . The same for  $u_w \in S_2$ , there is a  $\binom{f}{0} + \binom{f}{1} + \binom{f}{2}$  distinct representations of  $u_w \in S_2$ , and  $\binom{f}{0} + \binom{f}{1} + \binom{f}{2} - 1$  distinct representations for all  $u_w \in S_2$  with  $n = 3, 8$  and  $f = 2$ . Furthermore, let  $u_z \in S_j$  with  $u_z \in V(C_q) \setminus \{u_h, u_w\}$ ,  $1 \leq z \neq h \neq w \leq q$ ,  $j = 1, 2, \dots, f$ , and  $f = g - p$  then the representation of  $u_z$  with respect to  $\Pi$  is  $r(u_z|\Pi) = (1, 1, \dots, 1, 0, \dots)$ . Since the diameter of  $C_q + P_n$  graph is 2, then other than  $p$  position and the  $(p + 1)^{th}$  position of  $r(u_z|\Pi)$  can be filled by 1 or 2. There are at most two 1's and the rest filled by 2. If  $n = 2, 5, 12$  and  $p = 2$ , other than the  $p$  position and the  $(p + 1)^{th}$  position can be filled by 1 or 2 so that at most  $\binom{f-1}{0} + \binom{f-1}{1}$  distinct representation of  $u_z$  for  $n = 2, 5, 12$  and  $f = 2$ . For another  $p$  and  $f$ , the element of  $r(u_z|\Pi)$  other than  $p$  position and the  $(p + 1)^{th}$  position can be filled with at most two 1's and the rest filled by 2 so that there  $\binom{f-1}{0} + \binom{f-1}{1} + \binom{f-1}{2}$  distinct representations for  $u_z$ . The number of vertices element  $C_q$  is the number of vertex  $u_h, u_w$  and  $u_z$  in each class partition, then we obtain

$$\begin{aligned} |V(C_q)| &= |u_h, u_w \in S_i| + |u_z \in S_j| \\ &\leq ((\binom{f}{0} + \binom{f}{1} + \binom{f}{2}) - 1)(2) + ((\binom{f-1}{0} + \binom{f-1}{1}))(f) \\ q &\leq 2f^2 + f, \text{ for } n = 2, 5, 12 \text{ and } f = 2, \end{aligned}$$

$$\begin{aligned} |V(C_q)| &= |u_h, u_w \in S_i| + |u_z \in S_j| \\ &\leq ((\binom{f}{0} + \binom{f}{1} + \binom{f}{2}))(2) + ((\binom{f-1}{0} + \binom{f-1}{1} + \binom{f-1}{2}))(f) \\ q &\leq \frac{f^3 + f^2 + 4f + 4}{2}, \text{ for another } p, f. \end{aligned}$$

So, if  $v_b \in S_i$  for  $1 \leq i \leq p$ ,  $u_z \in V(C_q) \setminus \{u_h, u_w\}$  for  $1 \leq z \neq h \neq w \leq q$  contained in the partition class  $S_j$  for  $j = 1, 2, \dots, f$  with  $f = g - p$ , then  $q \leq 2f^2 + f$  for  $n = 2, 5, 12$  and  $f = 2$  and  $q \leq \frac{f^3 + f^2 + 4f + 4}{2}$  for another  $p, f$ .

**Lemma 2.7** Let  $\Pi = \{S_1, S_2, \dots, S_r, S_{r+1}, \dots, S_g\}$  be a resolving partition of  $C_q + P_n$  graph for  $q \geq 4$  and  $n \geq 2$ . If  $v_b \in S_i$  for  $i = 1, 2, \dots, r$  and some  $u_h \in V(C_q)$  for  $h \in [1, q]$ ,  $u_h \in S_1$ , then  $n \leq r + 1$  for  $r = 2$ ,  $n \leq \frac{r^2 + 3r - 2}{2}$  for  $r = 3$  and  $n \leq \frac{r^3 - 3r^2 + 6r - 2}{2}$  for  $r \geq 4$ .

**Proof.** Let  $\Pi = \{S_1, S_2, \dots, S_r, S_{r+1}, \dots, S_g\}$  be a resolving partition of  $C_q + P_n$  graph for  $q \geq 4$  and  $n \geq 2$ . Let  $v_b \in S_i$  for  $i = 1, 2, \dots, r$ ,  $v_c \in S_1$  with  $c \in [1, n]$  and  $u_h \in S_1$  with  $h \in [1, q]$ , we obtain the representation of  $v_c \in S_1$  with respect to  $\Pi$  is  $r(v_c|\Pi) = (0, \dots)$ , and the representation of  $u_h \in S_1$  with respect to  $\Pi$  is  $r(u_h|\Pi) = (0, 1, \dots, 1, \dots)$ . Since the diameter of  $C_q + P_n$  graph is 2, the elements of  $r(v_c|\Pi)$  other than the first position can be filled by 1 or 2. There are at most two 1's on representations other than the first position. Therefore,  $r - 1$  position on the representation of  $v_c \in S_1$  can be filled by two 1's and the rest can be 2. Let  $r = 2$ , there are at most  $\binom{r-1}{0} + \binom{r-1}{1}$  distinct representations of  $v_c \in S_1$ . Then let  $r \geq 3$ , there are at most  $\binom{r-1}{0} + \binom{r-1}{1} + \binom{r-1}{2}$  distinct representations of  $v_c \in S_1$ . Let  $v_d \in S_i$  with  $i = 2, 3, \dots, r$  and  $1 \leq d \neq c \leq n$ , we obtain  $r(v_d|\Pi) = (1, 0, \dots)$ . Because the diameter of  $C_q + P_n$  graph is 2, the element of  $r(v_d|\Pi)$  other than the first and second positions can be filled by 1 or 2. Let  $r = 2$ , there are at most  $\binom{r-2}{0}$  distinct representations of  $v_d \in S_i$ . Then let  $r = 3$ ,

there are at most  $\binom{r-2}{0} + \binom{r-2}{1}$  distinct representations of  $v_d \in S_i$ , and let  $r \geq 4$  there are at most  $\binom{r-2}{0} + \binom{r-2}{1} + \binom{r-2}{2}$  distinct representations of  $v_d \in S_i$ . Furthermore, because  $|P_n|$  is the sum of the number of vertices  $v_b$  in each class partition  $i$ , we obtain

(i) for  $r = 2$ ,

$$\begin{aligned} |V(P_n)| &= |v_c \in S_1| + |v_d \in S_2| \\ &\leq \binom{r-1}{0} + \binom{r-1}{1} + \binom{r-2}{0} \\ n &\leq r + 1, \end{aligned}$$

(ii) for  $r = 3$ ,

$$\begin{aligned} |V(P_n)| &= |v_c \in S_1| + |v_d \in S_i|, (i = 2, 3) \\ &\leq \binom{r-1}{0} + \binom{r-1}{1} + \binom{r-1}{2} + ((\binom{r-2}{0} + \binom{r-2}{1}))(2) \\ n &\leq \frac{r^2 + 3r - 2}{2}, \end{aligned}$$

(iii) for  $r \geq 4$ ,

$$\begin{aligned} |V(P_n)| &= |v_c \in S_1| + |S_i|, 2 \leq i \leq r \\ &\leq \binom{r-1}{0} + \binom{r-1}{1} + \binom{r-1}{2} + ((\binom{r-2}{0} + \binom{r-2}{1} + \binom{r-2}{2}))(r-1) \\ n &\leq \frac{r^3 - 3r^2 + 6r - 2}{2}. \end{aligned}$$

So, if  $v_b \in S_i$  for  $i = 1, 2, \dots, r$  and some  $u_h \in V(C_q)$  for  $h \in [1, q]$ ,  $u_h \in S_1$ , then  $n \leq r + 1$  for  $r = 2$ ,  $n \leq \frac{r^2 + 3r - 2}{2}$  for  $r = 3$  and  $n \leq \frac{r^3 - 3r^2 + 6r - 2}{2}$  for  $r \geq 4$ .

**Lemma 2.8** Let  $\Pi = \{S_1, S_2, \dots, S_r, S_{r+1}, \dots, S_g\}$  be a resolving partition of  $C_q + P_n$  graph for  $q \geq 4$  and  $n \geq 2$ . If vertex  $u_z \in V(C_q) \setminus \{u_h\}$  for  $1 \leq z \neq h \leq q$  contained in the partition class  $S_j$  for  $j = 1, 2, \dots, t$ ,  $t = g - r$ , then  $q \leq \frac{3t^2 + t}{2}$  for  $n = 3, 8$  and  $t = 2$  and  $q \leq \frac{t^3 + 3t + 2}{2}$  for another  $r, t$ .

**Proof.** Let  $\Pi = \{S_1, S_2, \dots, S_r, S_{r+1}, \dots, S_g\}$  be a resolving partition of  $C_q + P_n$  graph for  $q \geq 4$  and  $n \geq 2$ . Let  $v_b \in S_i$  for  $i = 1, 2, \dots, r$ ,  $u_h \in S_1$ ,  $u_z \in S_j$  for  $1 \leq h \neq z \leq q$ ,  $j = 1, 2, \dots, t$  with  $t = g - r$ . Based on Lemma 2.8, we obtain the representation of  $u_h \in S_1$  is  $r(u_h|\Pi) = (0, 1, \dots, 1, \dots)$ . Since the diameter of  $C_q + P_n$  graph is 2, the elements of  $r(u_h|\Pi)$  other than  $r$  positions can be filled by 1 or 2. There are at most two 1's on the representation other than  $r$  positions. Therefore,  $g - r$  position of representation  $u_h \in S_1$  can be filled by two 1's and the rest can be 2. There are at most  $\binom{g-r}{0} + \binom{g-r}{1} + \binom{g-r}{2}$  distinct representation of  $u_h \in S_1$  or at most  $\binom{t}{0} + \binom{t}{1} + \binom{t}{2}$ . Next, if  $v_b \in S_i$  for  $i = 1, 2, \dots, r$ , we obtain the representation of  $u_z \in S_j$  for  $1 \leq h \neq z \leq q$  and  $j = 1, 2, \dots, t$  with  $t = g - r$  is  $r(u_z|\Pi) = (1, \dots, 1, 0, \dots)$ . If  $n = 3, 8$  and  $t = 2$  then there are two equal representations, so as to  $n = 3, 8$  and  $t = 2$  there are at most  $\binom{t}{0} + \binom{t}{1} + \binom{t}{2} - 1$  distinct representation of  $u_h \in S_1$ . Since the diameter of  $C_q + P_n$  graph is 2, the elements of representation  $r(u_z|\Pi)$  other than  $r$  positions and the  $r + 1$ 's position can be filled by 1 or 2. If  $n = 3, 8$  and  $t = 2$ , there are at most  $\binom{g-r-1}{0} + \binom{g-r-1}{1}$  distinct representation of  $u_z \in S_j$  or at most  $\binom{t-1}{0} + \binom{t-1}{1}$  since  $t = g - r$ . Then, for another  $r, t$ , there are at most  $\binom{g-r-1}{0} + \binom{g-r-1}{1} + \binom{g-r-1}{2}$  or can be expressed by  $\binom{t-1}{0} + \binom{t-1}{1} + \binom{t-1}{2}$  distinct representation for  $u_z \in S_j$ . Since  $|V(C_q)|$  is the sum of the number of vertices  $u_h \in S_1$  and  $u_z \in S_j$ , then

$$\begin{aligned} |V(C_q)| &= |u_h \in S_1| + |S_j|, 1 \leq j \leq t \\ q &\leq ((\binom{t}{0} + \binom{t}{1} + \binom{t}{2}) - 1) + ((\binom{t-1}{0} + \binom{t-1}{1}))(t) \\ q &\leq \frac{3t^2 + t}{2}, \text{ for } n = 3, 8 \text{ and } t = 2 \end{aligned}$$

$$\begin{aligned}
|V(C_q)| &= |u_h \in S_1| + |S_j|, 1 \leq j \leq t \\
q &\leq \binom{t}{0} + \binom{t}{1} + \binom{t}{2} + ((\binom{t-1}{0}) + (\binom{t-1}{1}) + (\binom{t-1}{2}))(t) \\
q &\leq \frac{t^3 + 3t + 2}{2}, \text{ for another } r, t.
\end{aligned}$$

So, If vertex  $u_z \in V(C_q) \setminus \{u_h\}$  for  $1 \leq z \neq h \leq q$  contained in the partition class  $S_j$  for  $j = 1, 2, \dots, t$ ,  $t = g - r$ , then  $q \leq \frac{3t^2+t}{2}$  for  $n = 3, 8$  and  $t = 2$  and  $q \leq \frac{t^3+3t+2}{2}$  for another  $r, t$ .

**Lemma 2.9** Let  $\Pi = \{S_1, S_2, \dots, S_x, S_{x+1}, \dots, S_g\}$  be a resolving partition of  $C_q + P_n$  graph for  $q \geq 4$  and  $n \geq 2$ . If vertex  $v_b \in S_i$  for  $i = 1, 2, \dots, x$ , then  $n \leq \frac{x^3-x^2+2x}{2}$  for  $x \geq 2$ .

**Proof.** Let  $\Pi = \{S_1, S_2, \dots, S_x, S_{x+1}, \dots, S_g\}$  be a resolving partition of  $C_q + P_n$  graph for  $q \geq 4$  and  $n \geq 2$ . Let  $v_b \in S_i$  for  $i = 1, 2, \dots, x$  and  $u_a \in S_j$  for  $j = 1, 2, \dots, y$  with  $y = g - x$ . Representation of  $v_c \in S_1$  for  $1 \leq c \leq n$  is  $r(v_c|\Pi) = (0, \dots)$ . Since the diameter of  $C_q + P_n$  is 2, the elements of  $r(v_c|\Pi)$  other than first position can be filled with 1 or 2. There are  $x - 1$  position of the representation  $r(v_c|\Pi)$  can be filled at most two 1's and the rest can be filled by 2, so there are at most  $\binom{x-1}{0} + \binom{x-1}{1} + \binom{x-1}{2}$ . Since the number of  $P_n$  is  $n$ , then

$$\begin{aligned}
|V(P_n)| &= |S_i|, 1 \leq i \leq x \\
&\leq ((\binom{x-1}{0}) + (\binom{x-1}{1}) + (\binom{x-1}{2}))(x) \\
n &\leq \frac{x^3 - x^2 + 2x}{2}.
\end{aligned}$$

So, if vertex  $v_b \in S_i$  for  $i = 1, 2, \dots, x$  and  $u_a \in S_j$  for  $j = 1, 2, \dots, y$  with  $y = g - x$ , then  $q \leq \frac{x^3-x^2+2x}{2}$ .

**Lemma 2.10** Let  $\Pi = \{S_1, S_2, \dots, S_x, S_{x+1}, \dots, S_g\}$  be a resolving partition of  $C_q + P_n$  graph for  $q \geq 4$  and  $n \geq 2$ . If vertex  $u_a \in V(C_q)$  contained in partition class  $S_j$  for  $j = 1, 2, \dots, y$  with  $y = g - x$ , then  $q \leq \frac{y^3-y^2+2y}{2}$  for  $y \geq 2$ .

**Proof.** Let  $\Pi = \{S_1, S_2, \dots, S_x, S_{x+1}, \dots, S_g\}$  be a resolving partition of  $C_q + P_n$  graph for  $q \geq 4$  and  $n \geq 2$ . Let  $v_b \in S_i$  for  $i = 1, 2, \dots, x$  and  $u_a \in S_j$  for  $j = 1, 2, \dots, y$  with  $y = g - x$ . Representation of  $u_h \in S_y$  for  $1 \leq h \neq a \leq q$  is  $r(u_h|\Pi) = (1, \dots, 1, \dots, 0)$ . Since the diameter of  $C_q + P_n$  is 2, the elements of  $r(u_h|\Pi)$  other than last position can be filled with 1 or 2. There are at most two's 1 on the representation of  $u_h \in S_y$ . Therefore,  $y - 1$  position of the representation  $r(u_h|\Pi)$  can be filled at most two 1's and the rest can be filled by 2, so there are at most  $\binom{y-1}{0} + \binom{y-1}{1} + \binom{y-1}{2}$ . Since the number of  $C_q$  is  $q$ , then

$$\begin{aligned}
|V(C_q)| &= |S_j|, 1 \leq j \leq y \\
&\leq ((\binom{y-1}{0}) + (\binom{y-1}{1}) + (\binom{y-1}{2}))(y) \\
q &\leq \frac{y^3 - y^2 + 2y}{2}.
\end{aligned}$$

So, if vertex  $v_b \in S_i$  for  $i = 1, 2, \dots, x$  and  $u_a \in S_j$  for  $j = 1, 2, \dots, y$  with  $y = g - x$ , then  $q \leq \frac{y^3-y^2+2y}{2}$ .

**Theorem 2.11** Let  $C_q + P_n$  a graph with  $q \geq 4$  and  $n \geq 2$ ,  $\Pi = \{S_1, S_2, \dots, S_g\}$  be a resolving partition of  $C_q + P_n$ . The partition dimension of  $C_q + P_n$  is  $pd(C_q + P_n) = g$  with  $g = \min\{(p + f), (t + r), (x + y)\}$ , where  $p, f, r, t, x$  and  $y$  as stated in the Remark 2.4.

**Proof.** Let  $C_q + P_n$  be a graph with  $q \geq 4$  and  $n \geq 2$ , and let  $\Pi = \{S_1, S_2, \dots, S_g\}$  be a resolving partition of  $C_q + P_n$ . From Lemma 2.5 and Lemma 2.6 we have  $\Pi = \{S_1, S_2, \dots, S_p, S_{p+1}, \dots, S_g\}$  and  $g = p + f$  with  $p, f$  is the smallest positive integer such that  $n \leq 2$  for  $p = 2$ ,  $n \leq 2p - 1$  for  $p = 3$ ,  $n \leq p^2 - p$  for  $p = 4$  and  $n \leq \frac{p^3 - 5p^2 + 12p - 8}{2}$  for  $p \geq 5$  and  $q \leq 2f^2 + f$  for  $p = 2$  and  $f = 2$  and  $q \leq \frac{f^3 + f^2 + 4f + 4}{2}$  for another  $p, f$ . From Lemma 2.7 and Lemma 2.8 we have  $\Pi = \{S_1, S_2, \dots, S_r, S_{r+1}, \dots, S_g\}$  and  $g = r + t$  where  $r, t$  is the smallest positive integer such that  $n \leq r + 1$  for  $r = 2$ ,  $n \leq \frac{r^2 + 3r - 2}{2}$  for  $r = 3$  and  $n \leq \frac{r^3 - 3r^2 + 6r - 2}{2}$  for  $r \geq 4$  and  $q \leq \frac{3t^2 + t}{2}$  for  $r = 2$  and  $t = 2$  and  $q \leq \frac{t^3 + 3t + 2}{2}$  for another  $r, t$ . From Lemma 2.9 and Lemma 2.10 we have  $\Pi = \{S_1, S_2, \dots, S_x, S_{x+1}, \dots, S_g\}$  and  $g = x + y$  where  $x, y$  is the smallest positive integer such that  $n \leq \frac{x^3 - x^2 + 2x}{2}$  for  $x \geq 2$  and  $q \leq \frac{y^3 - y^2 + 2y}{2}$  for  $y \geq 2$ . Furthermore, because there are three different  $g$  values and partition dimension of graph is the minimum cardinality of the resolving partition, then  $pd(C_q + P_n) = \min\{(p + f), (r + t), (x + y)\}$ .  $\square$

### 3. Conclusion

Based on the discussion above, we conclude that the partition dimension of  $C_m + P_n$  graph for  $m \geq 3$  and  $n \geq 2$  can be divided into two theorems. The first theorem for  $C_3 + P_n$  with  $n \geq 2$  as stated in the Theorem 2.3 and the second theorem for  $C_q + P_n$  with  $q \geq 4$  and  $n \geq 2$  as stated in Theorem 2.11.

### 4. References

- [1] Chartrand, G., E. Salehi, and P. Zhang, *On the Partition Dimension of a Graph*, Congressus Numerantium, **131** (1998), 55-66.
- [2] Chartrand, G., E. Salehi, and P. Zhang, *The Partition Dimension of a Graph*, Aequation Math. **55**(2000), 45-54.
- [3] Chartrand, G., L. Eroh, M. Johnson, and O. Oellermann, *Resolvability in Graphs and the Metric Dimension of Graph*, Discrete Appl. Math. **105** (2000), 98-113.
- [4] Hidayat, D. W., *Dimensi Partisi pada Beberapa Kelas Graf*, Tugas akhir, FMIPA Universitas Sebelas Maret, Surakarta, 2015.
- [5] Tomescu, I., I. Javaid, and Slamin, *On The Partition Dimension and Connected Partition Dimension of Wheels*, Ars Combinatoria. **84** (2007), 311-318.