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H-Supermagic Labeling on Shrubs Graph and $L_m \odot P_n$

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Abstract. A finite simple graph G admits an H-covering if every edge of $E(G)$ belongs to a subgraph of G isomorphic to H. We said the graph $G = (V, E)$ that admits H-covering to be H-magic if there exists a bijection function $f : V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)| + |E(G)|\}$ such that for each subgraph H' of G isomorphic to $H, f(H') = \sum_{v \in V'} f(v) + \sum_{e \in E'} f(e) = m(f)$ is constant.
Furthermore, if $f(V) = 1, 2, \ldots, |V(G)|$ then G is called H-supermagic. In this research we defined $S_{2,2}$-supermagic labeling on shrub graph $\check{S}(m_1, m_2, \ldots, m_n)$ and fish-supermagic labeling on $L_m \odot P_n$ for $m, n \geq 2$.

1. Introduction

Gallian [2] defined a graph labeling as an assignment of integers to the vertices or edges, or both, subject to certain condition. Magic labeling is a type of graph labeling that the most often to be studied. In [2], magic labelings were first introduced in 1963 by Sedl*áč*ek.

We consider finite and simple graphs. The vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively. An edge-covering of G is a family of subgraphs $H_1, \ldots, H_k$ such that each edge of $E(G)$ belongs to at least one of the subgraphs $H_i, 1 \leq i \leq k$. If every H_i is isomorphic to a given graph H, then G admits and $H-$covering. An edge total labeling on G is a bijective function $f : V(G) \cup E(G) \to 1, 2, \ldots, |V(G)| + |E(G)|$, with the property that, given any edge xy, we have $f(x) + f(xy) + f(y) = k$.

In 2005, Gutiérrez and Lladó [4] generalized the concept of an edge-magic total labeling into an H-magic covering as follows. Let $G = (V, E)$ be a finite simple graph that admits H-covering. A bijection function $f : V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)| + |E(G)|\}$ is called H-magic labeling of G if for every subgraph $H' = (V', E')$ of G isomorphic to $H, f(H') = \sum_{v \in V'} f(v) + \sum_{e \in E'} f(e) = m(f)$ is constant. The graph G is called H-supermagic if $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$. In [4], it is proved that a complete bipartite graph $K_{n,n}$ is $K_{1,n}$-magic for $n \geq 1$.

Selvagopal and Jeyanthi in Gallian [2] proved that for a positive integer n, the k-polygonal snake of length n is C_4-supermagic; for $m \geq 2, n = 3$, or $n > 4$, $C_n \times P_m$ is C_4-supermagic; $P_2 \times P_n$ and $P_3 \times P_n$ are C_4-supermagic for all $n \geq 2$. Roswitha et al. [8] proved H-magic covering on some classes of graphs. In this research we investigate that a shrub graph $\check{S}(m_1, m_2, \ldots, m_n)$ admits a double star $S_{2,2}$-supermagic labeling and a $L_m \odot P_n$ admits a fish-supermagic labeling for $m, n \geq 2$.

2. Main Results

2.1. *k-balanced multiset*

Maryati et al. [6] defined a multiset as a set that allows the same elements in it. A multiset $\{a_1^{k_1}, a_2^{k_2}, a_3^{k_3}, ..., a_n^{k_n}\}$ is defined as a set of a_i as many as k_i for any integers n and $k_i, i \in [1, n]$. Let $V = \{a_1, a_2, a_3, a_4\}$ and $W = \{a_2, a_5, a_6\}$ be the multiset, then we have $V \biguplus W = \{a_1, a_2, a_2, a_3, a_4, a_5, a_6\}$. In [6] was introduced a technique of partitioning a multiset, called $k-$balanced multiset, as follows. Let $k \in N$ and Y be a multiset that contains positive integers. Y is said to be $k-$balanced if there exists k subsets of Y, say $Y_1, Y_2, \ldots, Y_k$, such that for every $i \in [i, k], |Y_i| = \frac{|Y|}{k}, \Sigma Y_i = \frac{\Sigma Y}{k} \in N$, and $\biguplus_{i=1}^{k} Y_i = Y$, then Y_i is called a balanced subset of Y.

Lemma 2.1 (Roswitha and Baskoro [7]) *Let x and y be non-negative integers. Let $X = [x+1, x(y+1)]$ with $|X| = xy$ and $Y = [x(y+2), 2x(y+1)-1]$ where $|Y| = xy$. Then, the multiset $K = X \uplus Y$ is xy-balanced with all its subsets are 2-sets.*

2.2. *(k, δ)-anti balanced multiset*

Inayah [5] defined (k, δ)-anti balanced multiset as follows. Let $k, \delta \in \mathbb{N}$ and X be a set containing the elements of positive integers. A multiset X is said to be (k, δ)-anti balanced if there exists k subsets from X, say $X_1, X_2, \ldots, X_k$ such that for every $i \in [1, k]$, $|X_i| = \frac{|X|}{k}$, $\biguplus_{i=1}^{k} X_i = X$, and for $i \in [1, k-1]$, $\sum X_{i+1} - \sum X_i = \delta$.

Here, we give several lemmas on (k, δ)-anti balanced multiset.

Lemma 2.2 Let x, y, z and k be non-negative integers, $k \geq 2$. Let $R = [x, x + \lfloor \frac{k}{2} \rfloor] \uplus [x+1, x + \lfloor \frac{k}{2} \rfloor] \uplus [y, y + \lfloor \frac{k-1}{2} \rfloor] \uplus [y, y + \lfloor \frac{k-1}{2} \rfloor - 1] \uplus [z, z + k - 1]$, then R is $(k, 2)$-anti balanced.

Proof.
For every $j \in [1, k]$ we define the multisets $R_j = \{x + \lfloor \frac{j}{2} \rfloor, y + \lfloor \frac{j-1}{2} \rfloor, z + (j-1)\}$. It is obvious that for each $j \in [1, k], |R_j| = 3, R_j \subset R$, and $\biguplus_{j=1}^{k} R_j = R$. Since $\sum R_j = x + y + z + \lfloor \frac{2j-1}{2} \rfloor + j - 1$ for every $j \in [1, k]$, then $\sum R_{j+1} - \sum R_j = 2$, R is $(k, 2)$-anti balanced.

□

Lemma 2.3 Let x and k be non-negative integers, $k \geq 2$. Let $Y = [x, x+k] \uplus [x+1, x+k-1]$, then Y is $(k, 2)$-anti balanced.

Proof.
For every $j \in [1, k]$ we define the mutisets $Y_j = \{x + j - 1, x + j\}$. Now we have for every $j \in [1, k], |Y_j| = 2, Y_j \subset Y$ and $\biguplus_{j=1}^{k} Y_j = Y$. Since $\sum Y_j = 2(x + j) - 1$ for every $j \in [1, k]$, then $\sum Y_{j+1} - \sum Y_j = 2$, Y is $(k, 2)$-anti balanced.

□

2.3. $S_{2,2}$*-supermagic Labeling on Shrubs Graphs.*

Maryati [6] defined a shrub graph $\check{S}(m_1, m_2, \ldots, m_n)$ as a graph that is obtained from a star graph $K_{1,n}$ for $n \geq 2$ with central vertex c, and adding some vertices and edges so that for every $i \in [1, n], v_i$ is related with the new $m_i \geq 1$ vertices. According to Grossman [3], a double star $S(n, m)$ is a graph consisting of the union of two stars $K_{1,n}$ and $K_{1,m}$ together with a line joining their centers.

In Figure 1 and Figure 2 we show examples of a shrub graph $\check{S}(m_1, m_2, \ldots, m_n)$ and a double star $S_{2,2}$.

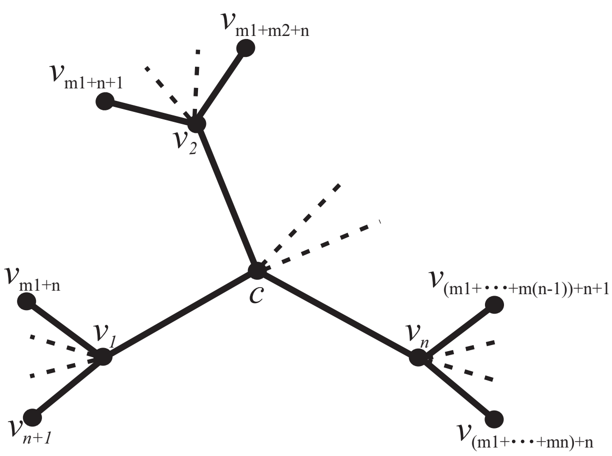

Figure 1. A shrub graph $\check{S}(m_1, m_2, \ldots, m_n)$.

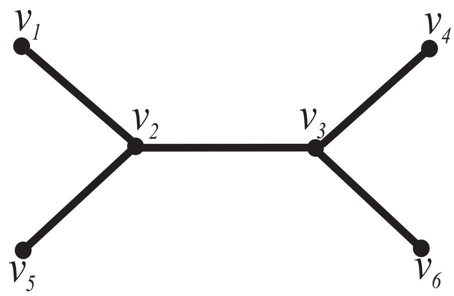

Figure 2. A double star $S_{2,2}$.

Theorem 2.4 *Any shrub graph $\check{S}(m_1, m_2, \ldots, m_n)$ is $S_{2,2}$-supermagic for any integer n and $m_i \geq 2, i \in [1, n]$.*

Proof.
Let G be a shrub graph $\check{S}(m_1, m_2, \ldots, m_n)$ for any integer n and $m_1, m_2, \ldots, m_n \geq 2$. Then $|V(G)| = m_1 + m_2 + \ldots + m_n + n + 1$ and $|E(G)| = m_1 + m_2 + \ldots + m_n + n$. Let $A = [1, 2(n + m_1 + m_2 + \ldots + m_n) + 1]$. Partition A into 3 sets, $A = K \cup X$, with $X = L \uplus M$ where $K = \{1\}, L = [2, n + 1 + m_1 + \ldots + m_n], M = [n + 2 + m_1 + \ldots + m_n, 2n + 1 + 2(m_1 + \ldots + m_n)]$. Let f be a total labeling of G and $s(f)$ be the supermagic sum in every subgraph H of G that is isomorphic to double star $S_{2,2}$.

Now we define a total labeling f on G as follows. Label the center vertex with 1. Apply Lemma 2.1 to partition X into $\{X_i, x \leq i \leq y\}$ with $x = 1$ and $y = n + m_1 + m_2 + \ldots + m_n$. We obtain that X is $n + m_1 + m_2 + \ldots + m_n$-balanced with all its subsets are 2-sets and $\sum X_i = 2(n + m_1 + m_2 + \ldots + m_n) + 3$. Use the smaller labels in every X_i for the vertices. Thus, the sum of labels of each double star $S_{2,2}$ is $s(f) = 10(n + m_1 + m_2 + \ldots + m_n) + 16$. Hence, the shrubs graph is $S_{2,2}$-supermagic.

□

The example of $S_{2,2}$-supermagic labeling on $\check{S}(3, 2, 4, 2)$ is shown in Figure 3.

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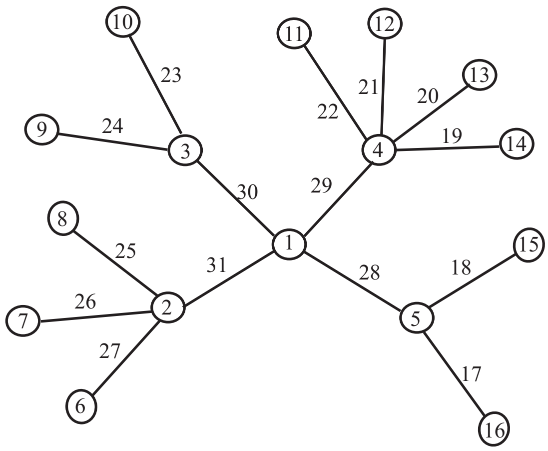

Figure 3. $S_{2,2}$-supermagic labeling on $\check{S}(3,2,4,2)$.

*2.4. **A Fish-Supermagic Labeling on** $L_m \odot P_n$.*

The corona $G_1 \odot G_2$ of two graphs G_1 and G_2 (where G_i has p_i points and q_i lines) is defined as the graph G obtained by taking one copy of G_1 and p_i copies of G_2, and then joining by a line i'th point of G_1 to every point in the i'th copy G_2.
$L_m \odot P_n$ graph is obtained by taking one copy of L_m and $2m$ copies of P_n and then joining by a line the i'th vertex of L_m to every vertex in the i'th copy of P_n.
Figure 4 is an example of $L_m \odot P_n$ graph.

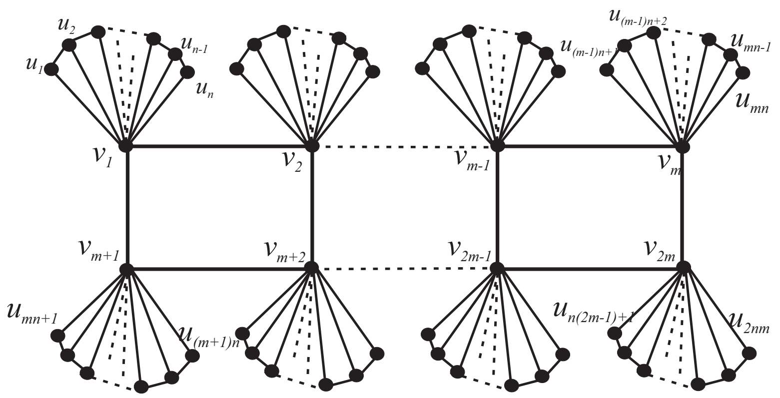

Figure 4. A $L_m \odot P_n$ graph.

Brandstadt [1] defined a fish graph as a graph on 6 vertices illustrated below.

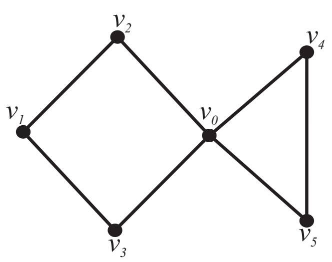

Figure 5. A fish graph.

Theorem 2.5 *Any $L_m \odot P_n$ graph for $m, n \geq 2$ is a fish-supermagic.*

Proof.
Let G be a $L_m \odot P_n$ graph for any integer $m, n \geq 2$. Then $|V(G)| = 2m(n+1)$ and

$|E(G)| = m(4n+1) - 2$. Let $A = [1, 2m(3n-1) + 5(m-1) + 3]$. We define a bijective function $f : V(G) \cup E(G) \to \{1, 2, \ldots, 2m(3n-1) + 5m - 2\}$. Partition A into 4 sets, $A = V \uplus W \uplus X \uplus Y$, where $V = [1, 2mn], W = [m(2n+3), m(4n+3) - 1], X = [m(4n+3), m(4n+3)+n-2] \biguplus [m(4n+3)+n, m(4n+3)+(2n-2)] \biguplus \ldots \biguplus [m(4n+3)+(2m-1)n, m(4n+3)+2mn-2]$, and $Y = [2mn+1, 2m(n+1)+(m-1)] \biguplus \{m(4n+3)+(n-1), m(4n+3)+(n-1)+n, m(4n+3)+(n-1)+2n, \ldots, m(4n+3)+(n-1)+2n(m-1)\}$.

Let f be a total labeling of G and $s(f)$ be the supermagic sum in every subgraph H of G that is isomorphic to fish graph. It is easier if a fish graph is divided into 2 subgraphs, $F_2 - \{v_0\}$ (a fan graph F_2 without a central vertex) and C_4.

There are two steps to label the $L_m \odot P_n$ graph. The first step is to prove that $2m(F_n - \{v_0\})$ is a $(F_2 - \{v_0\})$-supermagic using the elements of multisets $V \uplus W \uplus X$, and the second one is to prove that L_m is C_4-supermagic using the element of set Y.

Step 1. The first step is to prove that $2m(F_n - \{v_0\})$ is $(F_2 - \{v_0\})$-supermagic using the elements of multisets $V \uplus W \uplus X$. We consider two cases on this proof.

Case 1. $n = 2$. For $n = 2$, the number of $(F_2 - \{v_0\})$ subgraphs on graph G is $2m$ as shown in Figure 6.

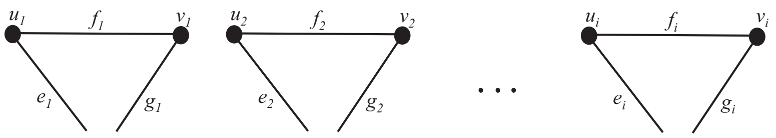

Figure 6. $F_2 - \{v_0\}$ subgraph on $L_m \odot P_2$.

Now, we prove that $2m(F_n - \{v_0\})$ on $L_m \odot P_2$ is $(F_2 - \{v_0\})$-supermagic using the element of multisets $V \uplus W \uplus X$. Let H be $2m(F_n - \{v_0\})$. We have $V(H) = \{u_i, v_i : 1 \le i \le 2m\}$ and $E(H) = \{e_i \cup f_i \cup g_i : 1 \le i \le 2m\}$. Let f be the total labeling of H and $s(f)$ be the supermagic sum in every $F_2 - \{v_0\}$ subgraph on $2m(F_n - \{v_0\})$. Then for every $1 \le i \le 2m$ we define

$$f(x) = \begin{cases} i, & \text{if } x = v_i, \\ 2m + i, & \text{if } x = u_i, \\ m(4n+3) + 1 - 2i, & \text{if } x = e_i, \\ m(4n+3) + 2(i-1), & \text{if } x = f_i, \\ m(4n+3) - 2i, & \text{if } x = g_i. \end{cases}$$

For every $1 \le i \le 2m$, let $(F_2 - \{v_0\})^i$ be the subgraph of $L_m \odot P_2$ with $V((F_2 - \{v_0\})^i) = \{u_i, v_i\}$ and $E((F_2 - \{v_0\})^i) = \{e_i, f_i, g_i\}$. It can be checked that for every $1 \le i \le 2m$, we have $\sum f((F_2 - \{v_0\})^i) = 3m(4n+3) + 2m - 1$. Hence $2m(F_2 - \{v_0\})$ is $(F_2 - \{v_0\})$-supermagic.

Case 2. $n > 2$. Now we prove that $2m(F_n - \{v_0\})$ is $(F_2 - \{v_0\})$-supermagic using the elements of multisets $V \uplus W \uplus X$. Let f be the total labeling of $2m(F_n - \{v_0\})$ and $s(f)$ be the supermagic sum in every $F_2 - \{v_0\}$ of $2m(F_n - \{v_0\})$. The total labeling of $(F_n - \{v_0\})^i$ for every $1 \le i \le 2m$ will be executed within two steps. Firstly, we label the subgraph of $(F_n - \{v_0\})^i$ called P_n^i, then after that we label the edges other than P_n^i that are incident with the vertices of P_n^i.

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Lemma 2.2 is applied for every i of P_n^i where $1 \leq i \leq 2m$ with $x_i = 1 + (i-1)\lceil\frac{n}{2}\rceil, y_i = 2m\lceil\frac{n}{2}\rceil + (i-1)\lfloor\frac{n}{2}\rfloor + 1, z = m(4n+3) + n(i-1)$, and $k = n-1$ using the element of $V \uplus X$. We have that $V \uplus X$ in every P_i is $(k,2)$-anti balanced.

Next, by applying Lemma 2.3 we label the edges other than P_n^i that are incident with the vertices of P_n^i using the elements of W with $x = m(4n+3) - ni$ and $k = n$, starting from the edge that incident with a vertex v_n until the edge that incident with a vertex v_1 in every P_n^i. Now we have that W for every $i \in [1, 2m]$ is $(k,2)$-anti balanced. This implies that multiset $V \uplus W \uplus X$ is $2km$-balanced. Hence $2m(F_n - \{v_0\})$ of $L_m \odot P_n$ is $(F_2 - \{v_0\})$-supermagic with $\sum f((F_2 - \{v_0\})^i) = m(12n + 2\lceil\frac{n}{2}\rceil + 2) + 7(m-1) + 6$. We conclude that for $n \geq 2, 2m(F_n - \{v_0\})$ is $(F_2 - \{v_0\})$-supermagic.

Step 2. The next step is to prove that L_m is C_4-supermagic using the element of set Y. Let $L_m \cong P_m \times P_2$ be a graph with $V(L_m) = \{u_i, v_i : 1 \leq i \leq m\}$ and $E(L_m) = \{u_iv_i : 1 \leq i \leq m\} \cup \{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq m-1\}$. Let f be a total labeling of $V(L_m) \cup E(L_m)$ using the element of set Y. The total labeling of L_m is defined as follows.

(i) Label each vertex of L_m as follows.

$$f(u_i) = m(2n+1) - (i-1), \quad \text{for } i \in [1, m].$$

$$f(v_i) = \begin{cases} m(2n+1) + \frac{i+1}{2}, & \text{for } i \text{ odd}, i \in [1, m], \\ m(2n+1) + \lceil\frac{m}{2}\rceil + \frac{i}{2}, & \text{for } i \text{ even}, i \in [1, m]. \end{cases}$$

(ii) Next, label each edge of L_m for every $i \in [1, m-1]$ as follows.

$$\begin{aligned} f(u_iu_{i+1}) &= m(4n+3) + ni - 1 \\ f(v_iv_{i+1}) &= m(2n+2) + i \end{aligned}$$

(iii) For every $i \in [1, m]$, label each edge of L_m as follows.

For m odd.

$$f(u_iv_i) = \begin{cases} m(4n+3) + n(\frac{m+1}{2}) + n(m-1) - n(\frac{i-1}{2}) - 1, & \text{for } i \text{ odd}, \\ m(4n+3) + mn + n(m-1) - n(\frac{i-2}{2}) - 1, & \text{for } i \text{ even}. \end{cases}$$

For m even.

$$f(u_iv_i) = \begin{cases} m(4n+3) + mn + n(m-1) - n(\frac{i-1}{2}) - 1, & \text{for } i \text{ odd}, \\ m(4n+3) + n(\frac{m}{2}) + n(m-1) - n(\frac{i-2}{2}) - 1, & \text{for } i \text{ even}. \end{cases}$$

For every $1 \leq i \leq m-1$, let C_4^i be as subgraph of $L_m \odot P_n$ with $V(C_4^i) = \{u_i, u_{i+1}, v_i, v_{i+1}\}$ and $E(C_4^i) = \{u_iu_{i+1}, v_iv_{i+1}, u_iv_i, u_{i+1}v_{i+1}\}$. It can be checked that for every $1 \leq i \leq m-1, \sum f(C_4^i) = m(22n+14) + n(2 + \lceil\frac{m+1}{2}\rceil) + \lceil\frac{m+1}{2}\rceil + (m-1)(3n+1)$. Hence L_m is C_4-supermagic.

It has been proved that $2m(F_n - \{v_0\})$ is $(F_2 - \{v_0\})$-supermagic and L_m is C_4-supermagic. Hence, $L_m \odot P_n$ is a fish-supermagic with its supermagic sum is $s(f) = m[(12n + 2 + 2\lceil\frac{n}{2}\rceil) + (22n+14)] + n(2 + \lceil\frac{m+1}{2}\rceil) + \lceil\frac{m+1}{2}\rceil + (m-1)(3n+8) + 6$. □

Figure 7 illustrates a fish-supermagic labeling on $L_3 \odot P_4$.

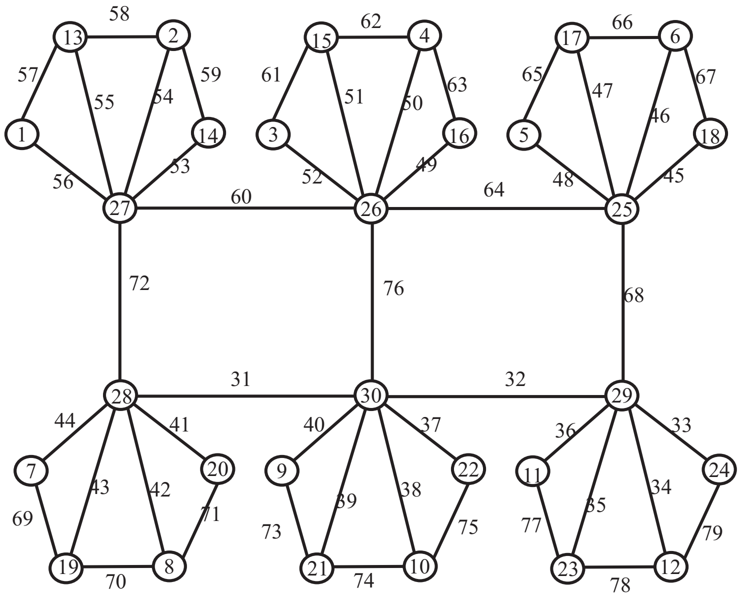

Figure 7. A fish-supermagic labeling on $L_3 \odot P_4$.

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